CHAPTER 6

EFFECT OF DELAYED IMPULSES ON FINITE-TIME STABILITY OF NONLINEAR SYSTEMS

6.1 Introduction[1]

Recent years have observed an increased interest in understanding impulsive dynamical systems that involve delays in the impulses, sometimes referred to as delayed impulses. These impulses refer to phenomena in which impulsive transients are affected by both present state and past state of the system. In many practical systems, time delays are unavoidable when sampling, processing, and transmitting impulsive information. In communication security systems based on impulsive synchronization [179], for example, a particular form of delay called sampling delays arise in impulsive transients due to the limited speed of signal sampling. Moreover, in some different models emerging from digital communication, neural networks, and ecological models, delayed impulses have found potential applications (see [180, 181, 182, 183, 184] and the references therein).

Inspired by the above discussions, we make an attempt to investigate the problem of nonlinear dynamical systems with delay-dependent impulses. The objective of this chapter is to construct finite time stability (FTS) criteria for such a system with stabilizing and destabilizing delayed impulses along with the derivation of impulse-dependent

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[1] The content of this chapter is under review in an international journal

settling-time functions. In the presence of stabilizing delayed impulses for FTS system, it has been demonstrated that a smaller settling-time bound can be attained. In addition, the class of impulsive sequences for which the system will be FTS is affected by the time delay of stabilizing impulses, where the effects are unfavorable for FTS analysis of the system. On the other hand, a larger bound for a settling-time function involving time delay can be attained when the system is subjected to destabilizing delayed impulses. The time delay for destabilizing impulses not only affects the class of impulsive sequences but also the settling-time function. The upper bound of time delay, up to which FTS of the system will be achieved, has been derived in this scientific contribution. This chapter can design a kind of impulse input in such a way that FTS can admit the desired settling-time bound under the influences of stabilizing and destabilizing delayed impulses.

6.2 Problem Formulation and Preliminaries

Suppose $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ represent the positive integers, real numbers and non-negative real numbers, respectively. $\mathbb{R}^n$ denotes the n-dimensional real linear space equipped with the Euclidean norm $|.|$. For any $M, N \subseteq \mathbb{R}^k (1 \leq k \leq n)$, $C(M,N) =\{ \psi : M \rightarrow N$ is continuous$\}$ and $PC(M,N) =\{\psi : M \rightarrow N$ has at most a finite number of jump discontinuities at point t, at which $\psi(t^+)$, $\psi(t^-)$ exist, and also it is continuous from the right at all points, i.e., $\psi(t^+) = \psi(t)\}$. For $\tau > 0$, $PC_\tau = PC([-\tau, 0], \mathbb{R}^n)$ with the norm defined by $||\varphi||_\tau = \sup\{\varphi(s), s \in [-\tau, 0]\}$. The class of continuous and strictly increasing functions $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to K if it satisfies $\gamma(0) = 0$; whereas it belongs to the class K_∞ if it is also radially unbounded.

We consider the general nonlinear impulsive system with impulsive delays of the form

$$\begin{cases} \dot{z}(t) = g(z(t)),\ t \geq t_0 \geq 0, & t \neq t_l, \\ z(t) = J(z(t^- - \tau_l)), & t = t_l,\ l \in \mathbb{Z}_+, \\ z_0 = \varphi(t_0 + s), & s \in [-\tau, 0], \end{cases} \tag{6.2.1}$$

where $z(t) \in \mathbb{R}^n$ is the system state, $\dot{z}(t)$ presents the right-hand derivative of $z(t)$, $z(t^+)$ and $z(t^-)$ denote the right limit and left hand limit of $z(t)$ at time t, respectively. We assume that $z(t^+) = z(t)$, i.e, the solutions of system (6.2.1) are right hand continuous. Function g :$\mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with $g(0) = 0$ and $J : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which satisfies $J(0) = 0$. The impulsive sequences $\{t_l, l \in \mathbb{Z}_+\}$(short for $\{t_l\}$) are some strictly increasing sequences on $\mathbb{R}_+$ satisfying $t_{l+1} - t_l \geq \tau_l$. We assume that $\lim_{l\to\infty} t_l = \infty$, which implies that no Zeno-type behaviour occur. Set Ω will be used to denote such impulse sequences for later use. The subset Ω_N of Ω is denoted as impulse sequences which satisfies $0 = t_0 < t_1 < t_2 < \ldots < t_N < \infty$, short for $\{t_l\}^N$, where N stands for the number of impulse points. The impulse input delay $\{\tau_l \geq 0,\ l \in \mathbb{N}\}$, and sequence $\{\tau_l\}$ is said to be the impulsive delay sequence which satisfies $\tau = \max \tau_l < \infty$, and function $\varphi \in E$ be the initial value, E is an open subset of PC_τ . The real number $t_0 = 0$ is the initial time, and for each $t \geq 0$, $z_t \in PC_\tau$ is defined by $z_t = z(t+s), s \in [-\tau, 0]$. Let us suppose that function g satisfies the relevant Lipschitz property, which states that the solution $z(t) = z(t, \varphi)$ with initial state $\varphi \in PC_\tau$ exists uniquely in forward time.

We also recall the following definitions and Lemma:

For locally Lipschitz continuous function $W : \mathbb{R}^n \to \mathbb{R}_+$ the upper right-hand Dini derivative is defined as

$$D^{+}W[z(t)]g(z(t)) = \limsup_{h\to 0^{+}} \frac{W(z(t)+hg(z(t))) - W(z(t))}{h}.$$

Definition 6.2.1. (See [71]) Let $E \subseteq PC([-\tau,0],\mathbb{R}^n)$ be an open set containing the origin. System (6.2.1) is said to be finite-time convergent for impulse sequence $\{t_l\} \in \Omega$, if there exists a function $T(\varphi,\{t_l\}) : E \times \Omega \to \mathbb{R}_+$ such that for initial state $\varphi \in E$, the solution $z(t,\varphi)$ of system (6.2.1) being subjected to the impulse $\{t_l\} \in \Omega$ is well defined and unique in forward time for $t \in [0, T(\varphi,\{t_l\}))$, and $z(t,\varphi) = 0$ for all $t \geq T(\varphi,\{t_l\})$. The system (6.2.1) is called FTS for the class Ω if it is Lyapunov stable and finite-time convergent for the class Ω. Moreover, the system (6.2.1) is globally FTS if $E = PC([-\tau,0],\mathbb{R}^n)$.

Lemma 6.2.1. (See [68]) Let $W : E \to \mathbb{R}^n$ is a locally Lipschitz continuous function, if there exist some constants $\mu > 0$, $0 < \nu < 1$, and two K-class functions V_1 and V_2 such that for all $z \in E$,

1. $V_1(|z(t)|) \leq W(z(t)) \leq V_2(|z(t)|)$,
2. $D^{+}W[z(t)]g(z(t)) \leq -\mu W^{\nu}(z(t))$,

hold, then the system (6.2.1) is FTS without impulses, and settling-time function with initial state $\varphi(0) \in E$ is estimated by

$$T(\varphi(0)) \leq T_0 = \frac{W^{1-\nu}(\varphi(0))}{\mu(1-\nu)}.$$

If $E = PC([-\tau,0],\mathbb{R}^n)$ and V is radially unbounded, then the system (6.2.1) is globally FTS.

6.3 Main Results

In this section, Lyapunov-like theorems of nonlinear systems for FTS are constructed with stabilizing and destabilizing delayed impulses, and estimate impulse and delay-dependent settling-time functions based on the impulsive point numbers and specific impulse moments. It is found that in the case of stabilizing impulses, a delay-dependent class of impulsive sequence exists in the FTS system. Whereas for destabilizing impulses, the class of impulsive sequences of FTS system as well as settling-time function is found to be delay-dependent.

Theorem 6.3.1. Let there exist two K-class functions V_1 and V_2, some constants $\mu > 0$, $\nu \in (0,1)$, $p \in (0,1)$, $q \in (p,1)$. For continuous and locally Lipschitz function $W : E \to \mathbb{R}^n$ such that for all $z \in E$, the following inequalities

1. $V_1(|z(t)|) \leq W(z(t)) \leq V_2(|z(t)|),$ (6.3.1)
2. $W(J(z(t_l))) \leq p^{\frac{1}{1-\nu}} W(J(t_l^- - \tau_l)), t = t_l,$ (6.3.2)
3. $D^+ W[z(t)] g(z) \leq -\mu W^{\nu}(z(t)),\ t \neq t_l,$ (6.3.3)

hold, then the system (6.2.1) with respect to impulse sequence $\{t_l\} \in \Omega_N$ is FTS and estimated settling-time function is given by

$$T(\varphi(0), \{t_l\}^N) \leq T_1 = q^N \frac{W^{1-\nu}(\varphi(0))}{\mu(1-\nu)}, \tag{6.3.4}$$

where $\{t_l\}^N \in \Omega_N$ satisfies

$$t_N \leq q^{N-1} \frac{(q-p) W^{1-\nu}(\varphi(0))}{\mu(1-\nu)(1-p)} - \frac{\tau p(1-p^N)}{(1-p)^2}, \tag{6.3.5}$$

and for FTS, the delay τ is estimated as

$$\tau < \frac{q^{N-1}(q-p)(1-p)}{p(1-p^N)}.$$

Moreover, if $E = PC([-\tau, 0], \mathbb{R}^n)$ and $V_1, V_2 \in K_\infty$, then the system (6.2.1) is globally FTS over impulse sequences set Ω.

Proof. For the prescribed impulse sequence $\Omega = \{t_l : l \in \mathbb{Z}_+\}$ and any given $\varphi \in E$, we denote the solution of system (6.2.1) by $z(t) = z(t, \varphi)$ from $(0, \varphi)$, where $\varphi(0) \neq 0$. According to Lemma 6.2.1, settling-time function without impulses is denoted as

$$\Gamma_{\varphi(0)} = \frac{W^{1-\nu}(\varphi(0))}{\mu(1-\nu)} > 0.$$

From equation (6.3.3), we have

$$\dot{W}(z(t)) \leq -\mu W^{\nu}(z(t)), \ \forall t \in [t_{l-1}, t_l). \tag{6.3.6}$$

Taking integration from both sides of the inequality (6.3.6), we get

$$\int\limits_{z(t_{l-1})}^{z(t)} \frac{dW(z(s))}{W^{\nu}(z(s))} \leq -\mu \int\limits_{t_{l-1}}^{t} ds,$$

$$i.e., \quad W^{1-\nu}(z(t)) \leq W^{1-\nu}(z(t_{l-1})) - \mu(1-\nu)(t - t_{l-1}), \ \forall t \in [t_{l-1}, t_l). \tag{6.3.7}$$

For $k = 1$, we have

$$W^{1-\nu}(z(t)) \leq W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t, \ \forall t \in [0, t_1).$$

When $t_1 \geq \Gamma_{\varphi(0)}$, the system solution $z(t)$ is free of impulse. In an instance, it is obvious that $W(z(t)) \leq (W(\varphi(0)) - \mu(1-\nu)t)^{\frac{1}{1-\nu}}$, $\forall t \in [0, \Gamma_{\varphi(0)})$, and $W(z(t)) \equiv 0$ for every $t \geq \Gamma_{\varphi(0)}$. Although $t < \Gamma_{\varphi(0)}$, suppose that the interval $[0, \Gamma_{\varphi(0)})$ contains n impulsive points, i.e., $0 < t_1 < \cdots < t_n < \Gamma_{\varphi(0)}$ for some $n \in \mathbb{Z}_+$. Given that $0 < p < 1$, it follows from $q < 1$ and equation (6.3.5) that $t_N < \Gamma_{\varphi(0)}$ and

$$\begin{aligned} t_j \leq t_N \leq & \frac{q^{N-1}(q-p)W^{1-\nu}(\varphi(0))}{\mu(1-\nu)(1-p)} - \frac{\tau p(1-p^N)}{(1-p)^2}, \\ = & \frac{q^j\left(1-\frac{p}{q}\right)}{1-p}\Gamma_{\varphi(0)} - \frac{\tau p(1-p^j)}{(1-p)^2}, \ j \in \mathit{\Lambda}, \end{aligned}$$

where we take $\mathit{\Lambda} = \{1, 2, 3, \ldots N\}$. The above inequality implies

$$pq^{j-1} + \frac{(1-p)t_j}{\Gamma_{\varphi(0)}} + \frac{\tau p(1-p^j)}{(1-p)\Gamma_{\varphi(0)}} \leq q^j, \ j \in \mathit{\Lambda}. \tag{6.3.8}$$

In view of equation (6.3.2), we obtain

$$\begin{aligned} W(z(t_1)) = W(J(z(t_1))) \leq & p^{\frac{1}{1-\nu}} W(z(t_1^- - \tau_1)), \\ W^{1-\nu}(z(t_1)) \leq & pW^{1-\nu}(z(t_1^- - \tau_1)) \\ = & p \begin{cases} W^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_1 - \tau_1 - t_0), & t_1 - \tau_1 > t_0, \\ W^{1-\nu}(\varphi(0)), & t_1 - \tau_1 \leq t_0, \end{cases} \\ W^{1-\nu}(z(t_1)) \leq & p\left(W^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_1 - \tau_1 - t_0)\right). \end{aligned} \tag{6.3.9}$$

Therefore, in view of equation (6.3.7), for all $t \in [t_1, t_2)$, we have

$$\begin{aligned} W^{1-\nu}(z(t)) \leq & W^{1-\nu}(z(t_1)) - \mu(1-\nu)(t - t_1) \\ \leq & pW^{1-\nu}(z(t^- - \tau_1)) - \mu(1-\nu)(t - t_1) \\ \leq & p\left(W^{1-\nu}(z(t_0)) - \mu(1-\nu)(t_1 - \tau_1 - t_0)\right) - \mu(1-\nu)(t - t_1) \\ = & pW^{1-\nu}(\varphi(0)) + \mu(1-p)(1-\nu)t_1 - \mu(1-\nu)t + \mu p(1-\upsilon)\tau_1. \end{aligned}$$

We know that

$$\Gamma_{\varphi(0)} = \frac{W^{1-\nu}(\varphi(0))}{\mu(1-\nu)} \Rightarrow \mu(1-\nu) = \frac{W^{1-\nu}(\varphi(0))}{\Gamma_{\varphi(0)}}.$$

In this way, we get

$$W^{1-\nu}(z(t)) \leq \left(p + \frac{(1-p)t_1}{\Gamma_{\varphi(0)}} + \frac{p\tau_1}{\Gamma_{\varphi(0)}}\right) W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t.$$

Combining the above with equation (6.3.8) and $\tau = \max \tau_l$, it can be derived that

$$W^{1-\nu}(z(t)) \leq q W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t, \ \forall t \in [t_1, t_2).$$

Thus, in view of equation (6.3.2), it follows that

$$\begin{aligned}
W(z(t_2)) = W(J(z(t_2)) &\leq p^{\frac{1}{1-\nu}} W(z(t_2^- - \tau_2)), \\
W^{1-\nu}(z(t_2)) &\leq p W^{1-\nu}(z(t_2^- - \tau_2)) \\
&= p \begin{cases} q W^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_2 - \tau_2), & t_2 - \tau_2 > t_1, \\ W^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_2 - \tau_2 - t_0), & t_1 \geq t_2 - \tau_2 > t_0, \\ W^{1-\nu}(\varphi(0)), & t_2 - \tau_2 \leq t_0, \end{cases} \\
W^{1-\nu}(z(t_2)) &\leq p \left[q W(\varphi(0)) - \mu(1-\nu)(t_2 - \tau_2) \right].
\end{aligned}$$

Then, in view of equation (6.3.7), for all $t \in [t_2, t_3)$, we have

$$
\begin{aligned}
W^{1-\nu}(z(t)) \leq & W^{1-\nu}(z(t_2)) - \mu(1-\nu)(t-t_2), \\
\leq & pW^{1-\nu}(z(t_2^- - \tau_2)) - \mu(1-\nu)(t-t_2) \\
\leq & p\left[qW^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_2-\tau_2)\right] - \mu(1-\nu)(t-t_2) \\
= & \left(pq + \frac{(1-p)t_2}{\Gamma_{\varphi(0)}} + \frac{p^2\tau_1}{\Gamma_{\varphi(0)}} + \frac{p\tau_2}{\Gamma_{\varphi(0)}}\right) W^{1-r}(\varphi(0)) - \mu(1-\nu)t.
\end{aligned}
$$

Combining the above with equation (6.3.8) and since $\tau = \max \tau_l$, it can be derived that

$$
W^{1-\nu}(z(t)) \leq q^2 W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t, \ \forall t \in [t_2, t_3).
$$

Similarly, by simple induction and using equation (6.3.8), we have the following inequality for all $j \in \varLambda$

$$
W^{1-\nu}(z(t)) \leq q^j W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t, \ \forall t \in [t_j, t_{j+1}), \tag{6.3.10}
$$

where t_{N+1} defines $q^N \Gamma_{\phi(0)}$. Note that

$$
t_{j+1} \leq \frac{q^j(q-p)W^{1-\nu}(\varphi(0))}{\mu(1-\nu)(1-p)} - \frac{\tau p(1-p^N)}{(1-p)^2} < q^N \Gamma_{\varphi(0)}, \forall j \leq N-1.
$$

It is observed from inequality (6.3.10) that the settling-time function cannot be defined in $[0, t_N)$. To find the settling-time function, set the right hand side of inequality (6.3.10) equal to zero on the interval $[t_N, t_{N+1}]$, and hence it follows that

$$
q^N W^{1-\nu}(\varphi(0)) - \mu(1-\nu)t = 0,
$$

$$t = q^N \Gamma_{\varphi(0)}.$$

Thus by the definition, we achieve that $W(z(t)) \leq W(\varphi(0)),\ t \geq 0$ and $W(z(t)) \equiv 0,\ \forall t \geq q^N \Gamma_{\varphi(0)}$. Therefore, if $\{t_l\}^N$ satisfy equation (6.3.5), then the system (6.2.1) is FTS and the estimated settling-time is defined by (6.3.4). Moreover, if $E = PC([-\tau, 0],\mathbb{R}^n)$ and $V_1, V_2 \in K_\infty$, then the system (6.2.1) is globally FTS. The proof is therefore complete.

Remark 6.3.1. Theorem 6.3.1 establishes certain sufficient criteria for FTS of nonlinear systems given by (6.2.1) including stabilizing delayed impulses, where the settling-time is influenced by the initial condition φ and the class Ω_N of impulse sequences. It is important to note that the settling-time T depends on last impulse time t_N, specified by parameters N, q. Observe that in impulses, the delay in time is fully accounted, i.e., when an impulse occurs at moment $t = t_l$, the state of system $x(t_l^+)$ is dependent on the historical state $x(t_l - \tau_l)$, where $\tau = \max \tau_l$.

Remark 6.3.2. FTS and estimated settling-time of nonlinear systems with impulses have been investigated in [71] where no delay effect is considered, i.e., when $\tau = 0$. However, in the above work, it has been discussed that if the system's state is subjected to impulses, then it is possible that the settling-time will be changed [71]. Therefore, it is important to get settling-time for delay-dependent impulses which is given in Theorem 6.3.1 of the present work.

Remark 6.3.3. In Theorem 6.3.1, FTS over a class of impulse criteria of the system (6.2.1) for stabilizing delayed impulses is estimated. For such case, the relationship between N, q, p and τ is proven to validate the influence of delayed impulses, when the delay τ is bounded by

$$\tau < \frac{q^N(q-p)(1-p)}{p(1-p^N)}. \tag{6.3.11}$$

This implies that if we take τ greater than the upper bound of the inequality (6.3.11) then the system's trajectory may not achieve the desired settling-time.

Remark 6.3.4. Theorem 6.3.1 shows that the effect of delay defined in the class of impulsive sequences for stabilizing impulses is distinct from the previous existing results as given in [71]. We obtain the same type of settling-time as derived in [71]. However, it is applicable for the different classes of impulsive sequences with delay where the settling-time is found to be independent of delay time parameter τ (see equation (6.3.4)) .

Corollary 6.3.1. Let $E_\epsilon = \{z \in \mathbb{R}^n : |\varphi(0)| \le \epsilon\}$ with $\epsilon > 0$ and let $T_d > 0$ be any constant, where T_d be a desired settling-time. Under the condition of Theorem 6.3.1, the estimated settling-time of system (6.2.1) $T(\varphi(0), \{t_l\}^N)$ satisfies

$$T((\varphi(0)), \{t_l\}^N) \le T_d,\ \forall \varphi(0) \in E_\epsilon,\ \forall \{t_l\}^N \in \Omega_N,$$

if N and $\{t_l\}^N$satisfy the following inequalities:

$$t_N \le \tfrac{q^{N-1}(q-p)V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)(1-p)} - \tfrac{\tau p(1-p^N)}{(1-p)^2},\ \ N \ge \log_q\left(\tfrac{T_d \mu(1-\nu)}{V_2^{1-\nu}(\epsilon)}\right).$$

where the delay τ is estimated as follows

$$\tau < \frac{q^N(q-p)(1-p)}{p(1-p^N)}.$$

Theorem 6.3.2. If there exist two K-class functions V_1 and V_2, some positive constants $\mu > 0$, $\nu \in (0,1)$, $p \in [1,\infty)$,$\epsilon > 0$ and for continuous and locally Lipschitz function $W : E \to \mathbb{R}^n$ such that for all $z \in E$, equations (6.3.1), (6.3.2) and (6.3.3) holds, then the system (6.2.1) is FTS over Ω_{N_0} which satisfies

$$\min\left\{j \in \mathbb{Z}_+ : t_j \geq p^{j-1}\frac{V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)} + \frac{\tau p(p^N-1)}{(p-1)^2}\right\} = N_0 < \infty. \tag{6.3.12}$$

Furthermore, the estimated settling-time function is given by

$$T(\varphi(0), \{t_l\}^{N_0}) \leq T_2 = \left(p^{N_0-1} + \frac{\tau p\left(p^{N_0-1}-1\right)}{(p-1)\Gamma_\epsilon}\right)\frac{V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)} \ \forall\varphi(0) \in E_\epsilon, \tag{6.3.13}$$

where N_0 is determined by the impulse sequence $\{t_l\}$.

Proof. For any given initial value $\varphi \in E_\epsilon$ and the impulse sequence $\Omega = \{t_l : l \in \mathbb{Z}_+\}$, we denote the solution of system (6.2.1) by $z(t) = z(t, \varphi)$ from $(0, \varphi)$, where $\varphi \neq 0$. Let us define

$$\Gamma_\epsilon = \frac{V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)} > 0.$$

It follows from equation (6.3.3) that

$$W^{1-\nu}(z(t)) \leq V_2^{1-\nu}(\epsilon) - \mu(1-\nu)t, \ \forall t \in [0, t_1).$$

When $t_1 \geq \Gamma_\epsilon$, then it follows that $N_0 = 1$ and $W(z(t)) \leq (V_2(\epsilon) - \mu(1-\nu)t)^{\frac{1}{1-\nu}}$, $\forall t \in [0, \Gamma_\epsilon)$, and $W(z(t)) \equiv 0$ for all $t \geq \Gamma_\epsilon$. Although $t_1 < \Gamma_\epsilon$, it implies that $N_0 \geq 2$. If $p \in [1, \infty)$, then we get by definition of Ω in equation (6.3.12) that $t_j < p^{j-1}\Gamma_\epsilon + \frac{\tau p(p^N-1)}{(p-1)^2}$ for $j \leq N_0 - 1$ and $t_{N_0} \geq p^{N_0-1}\Gamma_\epsilon + \frac{\tau p(p^N-1)}{(p-1)^2}$.

For $t \in [t_1, t_2)$, one can obtain

$$\begin{aligned} W^{1-\nu}(z(t)) \leq& W^{1-\nu}(z(t_1)) - \mu(1-\nu)(t-t_1) \\ \leq& p\left[W^{1-\nu}(\varphi(0)) - \mu(1-\nu)(t_1-\tau_1)\right] - \mu(1-\nu)(t-t_1), \\ W^{1-\nu}(z(t)) \leq& \left(p + \frac{p\tau_1}{\Gamma_\epsilon}\right)V_2^{1-\nu}(\epsilon) - \mu(1-\nu)t, \ \forall t \in [t_1, t_2). \end{aligned}$$

For $t \in [t_2, t_3)$, we get through similar analysis that

$$\begin{aligned} W^{1-\nu}(z(t)) \leq & W^{1-\nu}(z(t_2)) - \mu(1-\nu)(t-t_2) \\ \leq & p\left[\left(p + \frac{p\tau_1}{\Gamma_\epsilon}\right) V_2^{1-\nu}(\epsilon) - \mu(1-\nu)(t_2 - \tau_2)\right] - \mu(1-\nu)(t-t_2), \end{aligned}$$

$$W^{1-\nu}(z(t)) \leq \left(p^2 + \frac{p^2\tau_1}{\Gamma_\epsilon} + \frac{p\tau_2}{\Gamma_\epsilon}\right) V_2^{1-\nu}(\epsilon) - \mu(1-\nu)t, \ \forall t \in [t_2, t_3).$$

Similarly, we can finally find that

$$W^{1-\nu}(z(t)) \leq \left(p^{N_0-1} + \frac{1}{\Gamma_\epsilon}\sum_{l=1}^{j} p^{j-l+1}\tau_l\right) V_2^{1-\nu}(\epsilon) - \mu(1-\nu)t, \ t \in [t_{N_0-1}, t_{N_0}).$$

When $\tau = \max \tau_l$, it can be defined that

$$W^{1-\nu}(z(t)) \leq \left(p^{N_0-1} + \frac{\tau p\left(p^{N_0-1} - 1\right)}{(p-1)\Gamma_\epsilon}\right) V_2^{1-\nu}(\epsilon) - \mu(1-\nu)t, \ t \in [t_{N_0-1}, t_{N_0}).$$

Thus, we can derive that

$$W(z(t)) \leq \left(p^{N_0-1} + \frac{\tau p\left(p^{N_0-1} - 1\right)}{(p-1)\Gamma_\epsilon}\right)^{\frac{1}{1-\nu}} V_2(\epsilon),$$

for $t \in \left[0, \ \left(p^{N_0-1} + \frac{\tau p\left(p^{N_0-1}-1\right)}{(p-1)\Gamma_\epsilon}\right)\Gamma_\epsilon\right]$ and $W(z(t)) \equiv 0$ for all $t \geq \left(p^{N_0-1} + \frac{\tau p\left(p^{N_0-1}-1\right)}{(p-1)\Gamma_\epsilon}\right)\Gamma_\epsilon$. Therefore, if $\{t_l\}^{N_0}$ satisfies (6.3.12), then the system (6.2.1) is FTS and the settling-time function is bounded by (6.3.13). The proof is therefore completed.

Remark 6.3.5. Theorem 6.3.1 states that the stabilizing delayed impulses have no effect on the settling-time of the system (6.2.1) but does affect the class of impulsive sequences. However, Theorem 6.3.2 states that destabilizing impulses has an effect on the settling-time of the system (6.2.1) as well as on the class of impulsive sequences. The solution for destabilizing delayed impulses will take more time to reach the origin.

Remark 6.3.6. As compared to Theorem 6.3.1, it is found that the settling-time in case of destabilizing delayed impulses (as obtained in Theorem 6.3.2) is greater than that obtained in the case of stabilizing delayed impulses and it is also greater than the case of without any impulse in the system.

6.4 Numerical Simulations and Discussions

This section will consider two examples to carry out simulations in order to verify our proposed theoretical results as given in previous sections. The effects of delayed impulses on FTS and settling-time are shown from different aspects, by taking different parameters.

Example 6.4.1. Consider a 2-D nonlinear system with delayed impulses as given by

$$\begin{cases} \dot{z}_1 = z_2^2 - \sqrt{|z_1|}\operatorname{sgn}(z_1), & \\ \dot{z}_2 = -z_1 z_2 - \sqrt{|z_2|}\operatorname{sgn}(z_2), & t \geq 0, t \neq t_l, \\ \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.2 \\ -0.2 & 0.6 \end{pmatrix} \begin{pmatrix} z_1(t^- - \tau_l) \\ z_2(t^- - \tau_l) \end{pmatrix}, & t = t_l, l \in \mathbb{N} \\ x_{t_0} = \varphi(0), & \end{cases} \tag{6.4.1}$$

where $z = (z_1, z_2)^T$ is the system state and $t_0 = 0$. Let the initial value $\varphi(0) \in PC([-\tau, 0], \mathbb{R}^n)$ be given arbitrary. Taking a Lyapunov function $W(x) = |z|^2/2$, we get that equation (6.3.1) of Theorem 6.3.1 holds with $V_1(s) = V_2(s) = s^2/2$. It can be derived from a simple calculation that

$$D^+W[z(t)]g(z(t)) \leq -2^{\frac{3}{4}}W^{\frac{3}{4}}(z),\ t \neq t_l,$$

which means that equation (6.3.3) holds for $\mu = 2^{\frac{3}{4}}$ and $\nu = \frac{3}{4}$. Now, it derives that

$$W(J(z(t_l))) \leq 0.8^{\frac{1}{1-\frac{3}{4}}}W(z(t_l^- - \tau_l)),$$

which means that equation (6.3.2) holds for $p = 0.8$. It follows from Theorem 6.3.1 that the system (6.4.1) is globally FTS for every $\{t_l\}^N$ satisfying

$$t_N \leq q^{N-1}\frac{4(q-0.8)}{(0.2).2^{\frac{3}{4}}}\sqrt[4]{|\varphi|^2} - 20\tau(1-(0.8)^N),$$

where $q \in (0.8, 1)$, and the estimated settling-time function is given by

$$T(\varphi(0), \{t_l\}^N) \leq T_1 = q^N\frac{4\sqrt[4]{|\varphi|^2}}{2^{\frac{3}{4}}}.$$

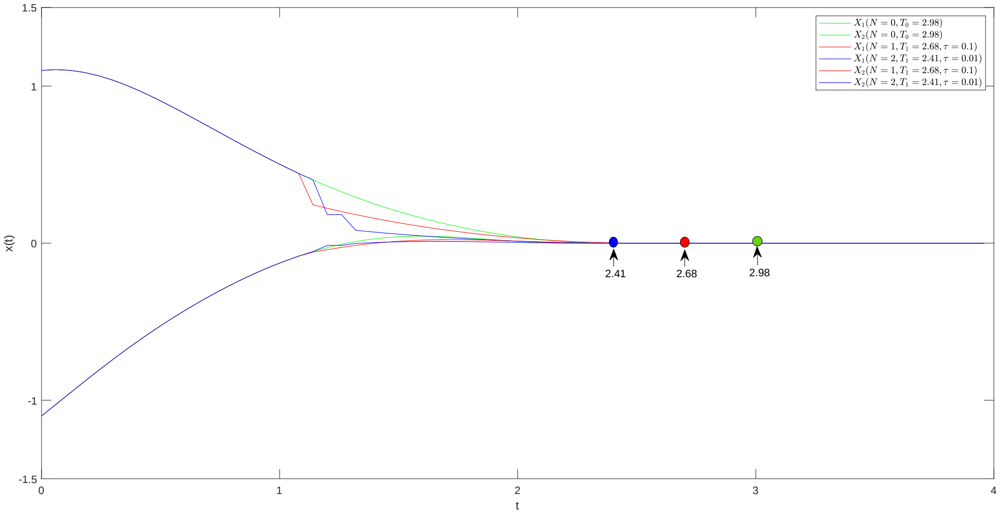

Figure 6.4.1: The state trajectories of system (5.3.8) with parameter $p = 0.8$ and initial state $\phi(0) = (1.1, -1.1)^T$.

First, we take $q = 0.9$, $\tau = \max \tau_l$, where $\tau_l = \frac{1}{10^l}$ and choose N to demonstrate the influence of the impulse jump numbers. Figure 6.4.1 shows the system state trajectories of (6.4.1) when $N = 0$ (without impulses), $N = 1$ with $\{t_l\}^1 = \{1.08\}$, and $N = 2$ with $\{t_l\}^2 = \{1.19, 1.26\}$, where the initial value $\varphi = \{1.1, -1.1\}$ is given. It can be viewed from the simulation that for $p < 1$, when we increase the value of N we get smaller settling-time function for the case of stabilizing impulses.

Example 6.4.2. Let us consider a simple ball motion with delayed impulses as follows:

$$\begin{aligned} \dot{v}(t) &= -v^m(t)(1 + v^2(t)), && t \neq t_l, \\ v(t) &= bw(t - \tau_l), && t = t_l, \end{aligned} \tag{6.4.2}$$

where the ball's velocity is $v(t)$ and initial velocity $v_0 = \epsilon$, a constant; $m \in (0, 1)$ is a constant; $b \in (0, 1)$ is the impulse strength, that is to say the ball's velocity will change abruptly at each moments $t = t_l$; $\{\tau_l; l \in \mathbb{Z}_+\}$ is the impulsive delay sequence. We take $\tau = \max \tau_l$, where $\tau_l = \frac{1}{10^l}$. Consider the Lyapunov function $W(v) = \frac{v^2}{2}$, and set $m = 0.6$ to examine the continuous dynamics of the model. Then one can get

$$D^+W[z(t)]g(z(t)) \leq -2^{\frac{4}{5}} W^{\frac{4}{5}}(z(t)), t \neq t_l,$$

which means that equation (6.3.3) holds for $\mu = 2^{\frac{4}{5}}$ and $\nu = \frac{4}{5}$. Then it follows from the results in Lemma 6.2.1 that the system (6.4.2) is FTS without impulses, and the estimated settling-time is defined by $T(\epsilon) \leq \frac{\epsilon^{1-m}}{(1-m)}$=$\Gamma_\epsilon$, i.e., for every initial velocity ϵ, the ball will stop before the time $t = \Gamma_\epsilon$. In numerical verification, assume $\epsilon = 1$. The velocity trajectory without impulse is shown in Figure 6.4.2 and the estimated settling-time (ball stopping time) satisfies $T(\epsilon) \leq 2.5$.

Further, we take the impulsive strength b and impulse times t_l so that the ball can be stopped before the desired time $t = T_d$, where $T_d \in (0, \mathit{\Gamma}_\epsilon)$ for the similar initial condition. It holds that

$$W(J(v(t_l))) = b^2 W(v(t_l^- - \tau_l)) \leq (b^{0.4})^{\frac{1}{1-0.8}} W(v(t_l^- - \tau_l)),$$

which means that equation (6.3.2) holds for $p = b^{0.4}$. Now, from the above Corollary 6.3.1, the ball's stopping time satisfies $T\left(\epsilon, \{t_l\}^N\right) \leq T_d$ when impulse sequence $\{t_l\}^N$will satisfy

$$t_N \leq \frac{q^{N-1}(q-p)V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)(1-p)} - \frac{\tau p(1-p^N)}{(1-p)^2}, \;\; N \geq \log_q \left(\frac{T_d \mu(1-\nu)}{V_2^{1-\nu}(\epsilon)} \right),$$

where $b < 1$ and $q \in (b^{0.4}, 1)$. For instance, set $b = 0.2$, $p = 0.2^{0.4}$ and $q = 0.6 \in (0.2^{0.4}, 1)$, if $T_d = 1.5 < 2.5$, then the class of impulses Ω_N satisfying $t_N \leq \frac{0.6^{N-1}}{2} - 2\tau(1-(0.5)^N)$, and for $N \geq 1$, the ball is stabilizing for such $T \leq T_d = 1.5$, which can be demonstrated in Figure 6.4.2 for $N = 1$ and the class of impulses $\Omega_1 = \{0.4\}$. Under the similar conditions, one can obtain the impulse control with Ω_N satisfying $t_N \leq \frac{0.6^{N-1}}{2} - 2\tau(1-(0.5)^N)$, and $N \geq 2$, the ball can stabilize such that $T_d = 0.9$, which is shown in Figure 6.4.2 when $N = 2$ and $\Omega_2 = \{0.18, 0.28\}$ are considered

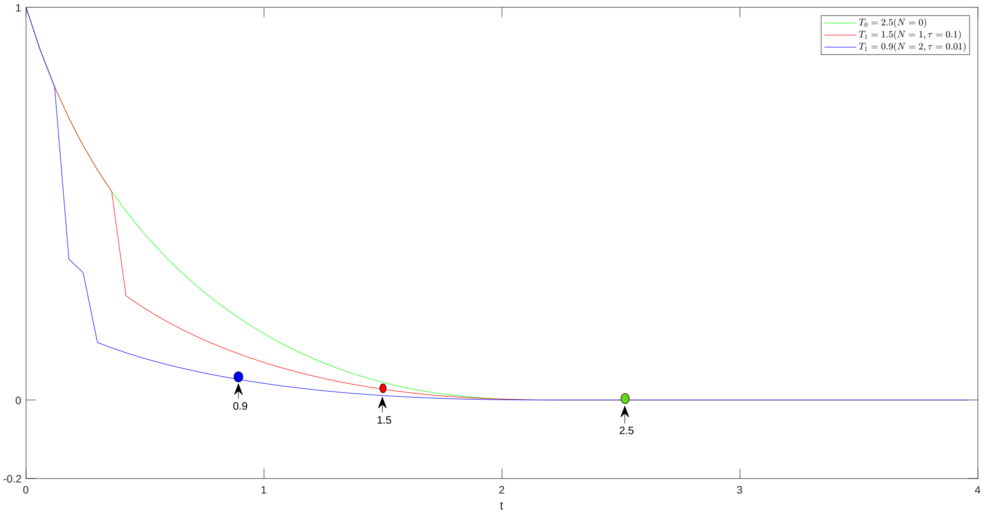

Figure 6.4.2: The state trajectories of system (5.3.9) with parameter $p = 0.6$.

On the other hand, if we wish to attain a stopping time bigger than T(i.e $T_d > T = 2.5$), a strength of impulse $b > 1$ can be taken that allows the ball to accelerate impulsively at each impulse moment $t = t_l$. It derives from Theorem 6.3.2 that the ball's stopping time satisfies $T(\epsilon, \{t_l\}) \leq T_d$ when the impulse time sequence $\{t_l\}^{N_0}$ satisfies

$$\min\left\{j \in \mathbb{Z}_+ : t_j \geq p^{j-1}\frac{V_2^{1-\nu}(\epsilon)}{\mu(1-\nu)} + +\frac{\tau p(p^N - 1)}{(p-1)^2}\right\} = N_0 < \infty.$$

In that case, the ball's stopping time will satisfy

$$T\left(\epsilon, \{t_l\}^{N_0}\right) \leq \left(p^{N_0-1} + \frac{\tau p\left(1 - p^{N_0-1}\right)}{(1-p)\Gamma_\epsilon}\right)\Gamma_\epsilon, \forall\{t_l\}^{N_0} \in \Omega_{N_0},$$

where $b > 1$ and $p = b^{0.4}$ are to be designed. For instance, we take $b = 1.3$, $p = 1.5$, and set $N_0 = 2$, then the ball's stopping time satisfies $T\left(\epsilon, \{t_l\}^{N_0}\right) \leq 3.9$ for impulsive control sequence $\{t_l\}^3 = \{1.9, 3, 7.5\ldots\} \in \Omega_{N_0}$. It can be seen from Figure 6.4.3.

Hence we can conclude from the above analysis and simulations that the desired ball's stop time bound T_d can be accomplished practically by using designed impulsive strength b and impulsive sequences $\{t_l\}^N$.

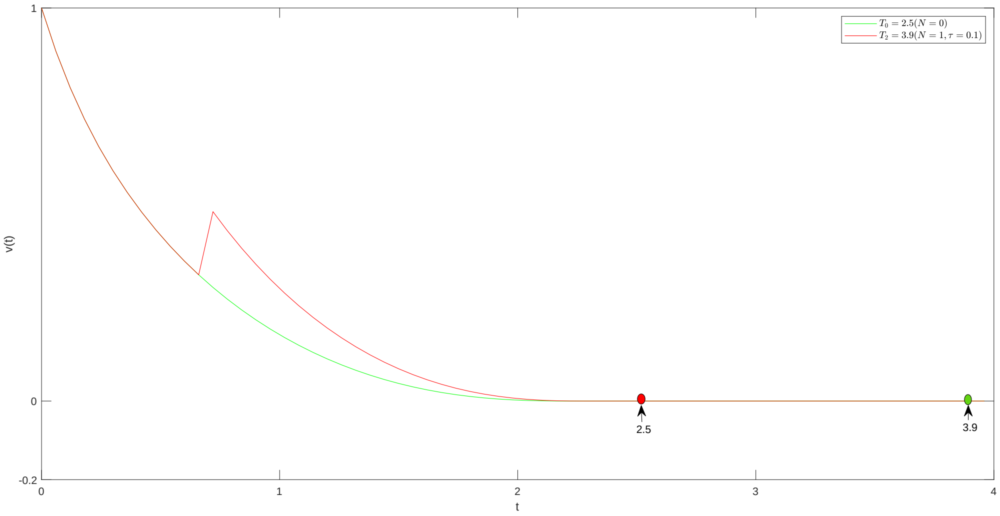

Figure 6.4.3: The state trajectories of system (5.3.9) with parameter $p = 1.5$.

6.5 Conclusion

The present work deals with the FTS of nonlinear impulsive systems under the influence of stabilizing and destabilizing delayed impulses. Based on Lyapunov theory, two FTS theorems have been proposed for the impulsive systems with delay-dependent impulses. Delay-dependent impulsive sequences have been designed for which the impulsive system will achieve the desired settling-time function. In the case of stabilizing impulses, time delay affects only the construction of impulsive sequences whereas it affects both impulsive sequences as well as settling-time functions in the case of destabilizing impulses. The upper bound of time delay has been estimated up to which the impulsive sequences are defined for the FTS systems. The effectiveness of the obtained results has been verified using two numerical examples. It is shown with the help of numerical examples that time-delay in impulsive control acts as a perturbation for the FTS systems.

The obtained results of this present chapter are based on the assumption that the inequality $t_{l+1} - t_l \geq \tau_l$ for all $l \in \mathbb{Z}_+$ always holds such that $z(t_l^+) = J(z(t_l - \tau_l))$,

$l \in \mathbb{Z}_+$. This implies that after hitting impulse moments t_l for $l \in \mathbb{Z}_+$, the solution of the FTS system will depend only on the solution between the impulsive moments t_{l-1} and t_l, while it is not always true for real world problems. Therefore, in future research work the drive will be given to investigate the FTS of nonlinear impulsive systems with delay-dependent impulses by relaxing this assumption on τ_l.