

# Chapter 6

## Design of Guidance Law for Planar Missile-Target System

### 6.1 Introduction

A guidance law is a set of rules or algorithms used to calculate the desired trajectory or path for a vehicle or object to follow [96]. These laws are commonly used in aerospace and defense applications, such as missiles, rockets, and spacecraft, to ensure that the vehicle reaches its intended target or destination with accuracy [125]. Designing high-precision guidance laws to intercept targets has become a crucial problem in real-world applications, especially when dealing with maneuvering targets. Control theory plays a crucial role in achieving precise target interception, and one of the main objectives has been to reduce of the Line-of-Sight (LOS) rate to zero or a small neighborhood of zero [97]. This chapter focuses on both maneuvering and non-maneuvering targets and discusses the following parts: (1) We discuss an adaptive super-twisting guidance law with an extended state observer. (2) We design an event-triggered adaptive super-twisting based guidance. (3) We design the guidance law with predefined-time convergence of the LOS rate.

The motivation for this work is drawn from [74], where an adaptive super-twisting sliding mode control law is proposed that considers unknown bounds of uncertainties and perturbations. Additionally, this work is motivated by an event-triggered-based adaptive super-twisting for attitude tracking of Reusable Launch Vehicles [136] and the design of controllers with arbitrary convergence time [37], where the convergence time can be chosen at the discretion of the designer.

This chapter makes the following contributions: (1) An extended state observer is used to estimate the unknown target acceleration. (2) An event-triggered adaptive super-twisting algorithm (ETASTA) guidance law is designed for intercepting a target with unknown disturbance. (3) The proposed guidance law does not require any information other than the existence of uncertainties/disturbances, and provisions are made to reduce the overestimated gains of the guidance law to uncertainties/disturbances. (4) It is demonstrated that the inter-event time is lower bounded by a positive value, which means that there is no accumulation of triggering events and no Zeno Phenomenon. (5) The LOS rate converges to zero within a predefined time, which can be chosen as per the designer's choice.

The rest of this chapter is organized as follows. Section 6.2 contains problem formulation of guidance law for planar missile-target engagement geometry. An adaptive super-twisting based guidance law with extended state observer is discussed in section 6.3. Section 6.4 presents an event-triggered adaptive super-twisting based guidance law. Section 6.5 introduces the guidance law with predefined-time convergence. Finally, section 6.6 summarizes the chapter.

## 6.2 Problem Formulation

In this part, the planar engagement geometry of missile and target is considered as shown in Figure 6.1. It represents the relative motion for missile and target denoted by  $M$  and  $T$  respectively. The missile-target relative range is  $r$ ,  $V_M$  and  $V_T$  are the velocities of the missile and target, and  $\phi_M$  and  $\phi_T$  are their flight-path angles and  $\theta$  is the LOS angle.

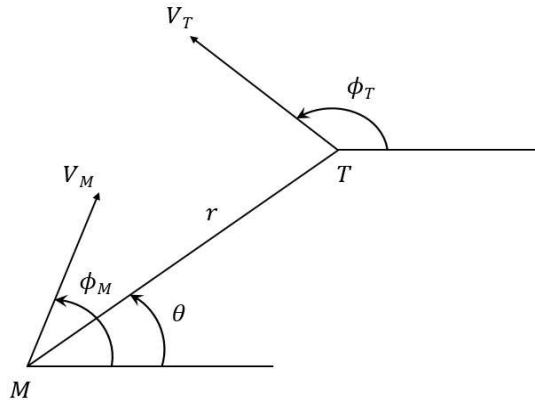


Figure 6.1: Planar missile-target engagement geometry

As it is clear from Figure 6.1, the kinematic equations of the engagement can be expressed as

$$\dot{r} = V_T \cos(\phi_T - \theta) - V_M \cos(\phi_M - \theta) \quad (6.1)$$

$$r\dot{\theta} = V_T \sin(\phi_T - \theta) - V_M \sin(\phi_M - \theta) \quad (6.2)$$

where  $\dot{\theta}$  is the LOS angular rate.

Differentiating the equations (6.1) and (6.2) with respect to time gives, respectively

$$\ddot{r} = r\dot{\theta}^2 + a_{Tr} - a_{Mr} \quad (6.3)$$

$$\ddot{\theta} = -\frac{2\dot{r}}{r}\dot{\theta} + \frac{a_{T\theta}}{r} - \frac{a_{M\theta}}{r}. \quad (6.4)$$

Denoting  $x_1 := \theta$ ,  $x_2 := \dot{\theta}$ , the above equations (6.4) can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{2\dot{r}}{r}x_2 + \frac{a_{T\theta}}{r} - \frac{u}{r} \end{aligned} \quad (6.5)$$

where  $u := a_{M\theta}$  is the acceleration component of the missile normal to the LOS axis and  $a_{T\theta}$  is the relative acceleration component of the target normal to the LOS axis. The target acceleration  $a_{T\theta}$  is unknown but it is assumed to be bounded such as  $|a_{T\theta}| \leq a_{T\theta}^{\max}$ ,  $a_{T\theta}^{\max} > 0$ ,  $\forall t \geq 0$ .

## 6.3 Adaptive Super-Twisting Guidance Law with Extended State Observer

### 6.3.1 Adaptive Super-Twisting Guidance Law

The unknown target acceleration is viewed as uncertainty and the AST guidance law is proposed for the resulting nominal system. The objective is to design a guidance law using STA to achieve the target interception in finite time avoiding over-estimation of the gain values.

Consider the following form of non-linear dynamical system

$$\dot{x} = f(x) + h(x)u \quad (6.6)$$

where  $x \in \mathbb{R}^n$  is the system state,  $u \in \mathbb{R}$  is the control input and  $f(x)$  is a non-linear function.

**Assumption 9** A sliding surface  $s(x, t) \in \mathbb{R}$  is designed in such a way as to achieve the system dynamics (6.6) on the sliding surface  $s(x, t) = 0$ .

**Assumption 10** The relative degree of the system (6.6) with the defined sliding surface  $s(x, t)$  with respect to control  $u$  equals one and the internal dynamics are stable.

Thus, the dynamics from input to output becomes

$$\begin{aligned}\dot{s} &= \underbrace{\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} f(x)}_{\phi(x,t)} + \underbrace{\frac{\partial s}{\partial x} h(x)}_{b(x,t)} u \\ &= \phi(x, t) + b(x, t)u.\end{aligned}\tag{6.7}$$

**Assumption 11** The function  $\phi(x, t) \in \mathbb{R}$  is bounded and  $b(x, t) \in \mathbb{R}$  is known with  $b(x, t) \neq 0 \forall x$  and  $t \in [0 \infty)$ .

For the system (6.5), the sliding surface is

$$s = x_2 + k|x_1|^\eta \text{sign}(x_1)\tag{6.8}$$

where  $k$  is positive constant and  $0.5 < \eta < 1$ .

Suppose that  $r, \dot{r}, x_1$ , and  $x_2$  can be measured.

The AST guidance law is designed as:

$$\begin{aligned}u &= crx_2 - 2\dot{r}x_2 + r\alpha|s|^{1/2}\text{sign}(s) - rv \\ \dot{v} &= -\frac{\beta}{2}\text{sign}(s)\end{aligned}\tag{6.9}$$

where  $c$  is positive constant and the adaptive gains are the functions  $\alpha(s, \dot{s}, t), \beta(s, \dot{s}, t)$  which are updated by the adaptive laws [74]:

$$\begin{aligned}\dot{\alpha} &= \begin{cases} w_1 \sqrt{\frac{\gamma_1}{2}} \text{sign}(|s| - \mu), & \text{if } \alpha > \alpha_m \\ \zeta, & \text{if } \alpha \leq \alpha_m \end{cases} \\ \beta &= 2\epsilon\alpha\end{aligned}\tag{6.10}$$

where  $w_1, \gamma_1, \mu, \zeta$  and  $\epsilon$  are arbitrary positive constants. The quantity  $\alpha_m$  is an arbitrary small positive constant.

**Theorem 5** Consider the nonlinear system (6.5), which satisfies the assumption  $|a_{T\theta}| \leq a_{T\theta}^{max}$  with sliding surface (6.8). Then, for any initial condition  $x(0) = [x_1(0), x_2(0)]^T$ , there exists  $\mu$  and a finite time  $t_f > 0$  in which the states of (6.5) converge to the origin with the designed AST guidance law (6.9) with adaptive gains (6.10).

*Proof:* Consider  $x_1$  and  $s$  to be non-zero at the initial time. The time derivative of (6.8) is

$$\begin{aligned}\dot{s} &= \dot{x}_2 + k\eta|x_1|^{\eta-1}x_2 \\ &= -\frac{2\dot{r}}{r}x_2 + \frac{a_{T\theta}}{r} - \frac{u}{r} + k\eta|x_1|^{\eta-1}x_2.\end{aligned}\quad (6.11)$$

Substituting (6.9) into (6.11) yields

$$\begin{aligned}\dot{s} &= \psi(x, t) + g(t) - \alpha|s|^{1/2}\text{sign}(s) + v \\ \dot{v} &= -\frac{\beta}{2}\text{sign}(s)\end{aligned}\quad (6.12)$$

where  $\psi(x, t) = -cx_2 + \sigma(x_1)x_2$  with  $\sigma(x_1) = \eta k|x_1|^{\eta-1}$  and  $g(t) = a_{T\theta}/r$ . To avoid the singularity,  $\sigma(x_1) = 0$  if  $x_1 = 0$  and  $s \neq 0$ , otherwise,  $\sigma(x_1) = \eta k|x_1|^{\eta-1}$ . Considering the missile and the target to have significant sizes of air-frame, they can intercept if  $r$  becomes sufficiently small. Thus, it is justified to assume that the derivative of  $g(t)$  remains bounded as

$$0 < |\dot{g}(t)| \leq \gamma \quad (6.13)$$

where  $\gamma$  is bounded constant.

Suppose  $v_\star = v + g(t)$ . Then, (6.12) becomes,

$$\begin{aligned}\dot{s} &= -\alpha|s|^{1/2}\text{sign}(s) + v_\star + \psi(x, t) \\ \dot{v}_\star &= -\frac{\beta}{2}\text{sign}(s) + \dot{g}(t).\end{aligned}\quad (6.14)$$

Let us consider new state variables are defined as

$$z = [z_1 \ z_2]^T = [|s|^{1/2}\text{sign}(s) \ v_\star]^T \quad (6.15)$$

From the system (6.12), assume that the function  $\psi(x, t)$  remains bounded as

$$|\psi(x, t)| \leq \delta|s|^{1/2}. \quad (6.16)$$

Considering (6.13) and (6.16), one can write

$$\begin{aligned}\psi(x, t) &= \Delta_1(x, t)|s|^{1/2}\text{sign}(s) = \Delta_1(x, t)z_1 \\ \dot{g}(t) &= \frac{\Delta_2(x, t)}{2}\text{sign}(s) = \frac{\Delta_2(x, t)}{2} \frac{z_1}{|z_1|}\end{aligned}\quad (6.17)$$

In view of (6.13) and  $z_1 = |s|^{1/2}\text{sign}(s) \Rightarrow |z_1| = |s|^{1/2}$ , one can represent (6.17) as

$$0 < \Delta_1(x, t) \leq \delta, \quad 0 < \Delta_2(x, t) \leq 2\gamma.$$

The equivalent representation of system (6.14) becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = A(z_1) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (6.18)$$

where

$$A(z_1) = \frac{1}{2|z_1|} \begin{bmatrix} -(\alpha - \Delta_1(x, t)) & 1 \\ -(\beta - \Delta_2(x, t)) & 0 \end{bmatrix}. \quad (6.19)$$

Consider the Lyapunov function

$$V(z, \alpha, \beta) = V_0(z) + \frac{1}{2\gamma_1}(\alpha - \alpha^*)^2 + \frac{1}{2\gamma_2}(\beta - \beta^*)^2 \quad (6.20)$$

where  $\gamma_1, \gamma_2, \alpha^* > 0$  and  $\beta^* > 0$  are positive constants and

$$V_0(z) = z^T P z; \quad P = \begin{bmatrix} \lambda + 4\epsilon^2 & -2\epsilon \\ -2\epsilon & 1 \end{bmatrix}, \quad \lambda > 0, \quad \epsilon > 0$$

and  $P$  is a positive-definite matrix.

Taking the time derivative of (6.20),

$$\dot{V}(z, \alpha, \beta) = \dot{V}_0(z) + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha} + \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta}. \quad (6.21)$$

In view of the system (6.18), the first term of (6.21) can be presented as

$$\dot{V}_0(z) = z^T [A^T P + P A] z \leq -\frac{1}{2|z_1|} z^T Q z. \quad (6.22)$$

The symmetric matrix  $Q$  for the system (6.14) is

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & 4\epsilon \end{bmatrix} \quad (6.23)$$

where

$$Q_{11} = 2\lambda\alpha + 4\epsilon(2\epsilon\alpha - \beta) - 2(\lambda + 4\epsilon^2)\Delta_1(x, t) + 4\epsilon\Delta_2(x, t)$$

$$Q_{12} = Q_{21} = (\beta - 2\epsilon\alpha - \lambda - 4\epsilon^2) + 2\epsilon\Delta_1(x, t) - \Delta_2(x, t)$$

As  $\beta = 2\epsilon\alpha$ ,  $Q$  is positive-definite with a minimal eigenvalue  $2\epsilon \leq \lambda_{\min}(Q)$  if

$$\alpha > \frac{\delta(\lambda + 4\epsilon^2) - \epsilon(4\gamma + 1)}{\lambda} + \frac{(2\epsilon\delta - 2\gamma - \lambda - 4\epsilon^2)^2}{12\epsilon\lambda}. \quad (6.24)$$

By recalling Rayleigh inequality

$$\lambda_{\min}(P)\|z\|^2 \leq z^T P z \leq \lambda_{\max}(P)\|z\|^2. \quad (6.25)$$

Denote

$$\begin{aligned} \|z\|^2 &= z_1^2 + z_2^2 = |s| + z_2^2 \Rightarrow |s|^{1/2} \leq \|z\| \\ \text{and } |z_1| &= |s|^{1/2} \leq \|z\| \leq \frac{V_0^{1/2}(z)}{\lambda_{\min}^{1/2}(P)}. \end{aligned} \quad (6.26)$$

From (6.25) and as  $2\epsilon \leq \lambda_{\min}(Q)$ , one can write (6.22) as

$$\dot{V}_0(z) \leq -\frac{\epsilon}{|z_1|} \frac{V_0(z)}{\lambda_{\max}(P)}. \quad (6.27)$$

Substituting (6.26) into (6.27) gives

$$\dot{V}_0(z) \leq -\tau V_0^{1/2}(z), \quad \tau = \frac{\epsilon \lambda_{\min}^{1/2}(P)}{\lambda_{\max}(P)}. \quad (6.28)$$

In accordance with (6.28), one can represent (6.21) as

$$\begin{aligned} \dot{V}(z, \alpha, \beta) &\leq -\tau V_0^{1/2}(z) + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha} + \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} \\ &= -\tau V_0^{1/2}(z) - \frac{w_1}{\sqrt{2\gamma_1}}|\alpha - \alpha^*| - \frac{w_2}{\sqrt{2\gamma_2}}|\beta - \beta^*| \\ &\quad + \frac{1}{\gamma_1}\dot{\alpha}(\alpha - \alpha^*) + \frac{w_1}{\sqrt{2\gamma_1}}|\alpha - \alpha^*| + \frac{1}{\gamma_2}\dot{\beta}(\beta - \beta^*) + \frac{w_2}{\sqrt{2\gamma_2}}|\beta - \beta^*|. \end{aligned} \quad (6.29)$$

Using the inequality

$$(x^2 + y^2 + z^2)^{1/2} \leq |x| + |y| + |z|$$

and from (6.20), one obtains

$$-\tau V_0^{1/2}(z) - \frac{w_1}{\sqrt{2\gamma_1}}|\alpha - \alpha^*| - \frac{w_2}{\sqrt{2\gamma_2}}|\beta - \beta^*| \leq -\tau_0 V^{1/2}(z, \alpha, \beta) \quad (6.30)$$

with  $\tau_0 = \min(\tau, w_1, w_2)$ . From (6.30), one can rewrite (6.29) as

$$\begin{aligned} \dot{V}(z, \alpha, \beta) &\leq -\tau_0 V^{1/2}(z, \alpha, \beta) + \frac{1}{\gamma_1}\dot{\alpha}(\alpha - \alpha^*) + \frac{1}{\gamma_2}\dot{\beta}(\beta - \beta^*) \\ &\quad + \frac{w_1}{\sqrt{2\gamma_1}}|\alpha - \alpha^*| + \frac{w_2}{\sqrt{2\gamma_2}}|\beta - \beta^*|. \end{aligned} \quad (6.31)$$

Suppose the adaptive law (6.10) makes the gains  $\alpha(t)$  and  $\beta(t)$  bounded. Then, there exist positive constants  $\alpha^*$ ,  $\beta^*$  such that  $\alpha(t) - \alpha^* < 0$  and  $\beta(t) - \beta^* < 0$ , for all  $t \geq 0$ . Now, the Eq. (6.31) can be written as

$$\begin{aligned}\dot{V}(z, \alpha, \beta) &\leq -\tau_0 V^{1/2}(z, \alpha, \beta) - |\alpha - \alpha^*| \left( \frac{1}{\gamma_1} \dot{\alpha} - \frac{w_1}{\sqrt{2\gamma_1}} \right) - |\beta - \beta^*| \left( \frac{1}{\gamma_2} \dot{\beta} - \frac{w_2}{\sqrt{2\gamma_2}} \right) \\ \dot{V}(z, \alpha, \beta) &\leq -\tau_0 V^{1/2}(z, \alpha, \beta) + \Xi\end{aligned}\quad (6.32)$$

where  $\Xi = -|\alpha - \alpha^*| \left( \frac{1}{\gamma_1} \dot{\alpha} - \frac{w_1}{\sqrt{2\gamma_1}} \right) - |\beta - \beta^*| \left( \frac{1}{\gamma_2} \dot{\beta} - \frac{w_2}{\sqrt{2\gamma_2}} \right)$ .

Suppose  $|s| > \mu$  and  $\alpha(t) > \alpha_m$ ,  $\forall t \geq 0$ . After selecting  $\epsilon = \frac{w_2}{2w_1} \sqrt{\gamma_2/\gamma_1}$ , in view of (6.10),  $\dot{\alpha} = w_1 \sqrt{\gamma_1/2}$  and  $\dot{\beta} = w_2 \sqrt{\gamma_2/2}$ . Then, the Eq. (6.32) reduces to

$$\dot{V}(z, \alpha, \beta) \leq -\tau_0 V^{1/2}(z, \alpha, \beta).\quad (6.33)$$

Let  $|s| \leq \mu$ . Then,  $\alpha(t)$  converges in view of (6.10) as

$$\dot{\alpha} = \begin{cases} -w_1 \sqrt{\frac{\gamma_1}{2}}, & \text{if } \alpha > \alpha_m \\ \zeta, & \text{if } \alpha \leq \alpha_m \end{cases}\quad (6.34)$$

and the term

$$\Xi = \begin{cases} 2|\alpha - \alpha^*| \frac{w_1}{\sqrt{2\gamma_1}}, & \text{if } \alpha > \alpha_m \\ -|\alpha_m - \alpha^* + \zeta \cdot t| \left( \frac{\zeta}{\gamma_1} - \frac{w_1}{\sqrt{2\gamma_1}} \right), & \text{if } \alpha \leq \alpha_m. \end{cases}\quad (6.35)$$

As per the above discussion, the proposed AST guidance law forces the sliding surface as well as its derivative to certain domains in finite time under uncertainties. Therefore, the missile-target engagement occurs in finite time. ■

**Remark 13** *By utilizing the adaptive super-twisting algorithm, the guidance law compels the sliding surface and its derivative to converge within finite time to specific domains, even in the presence of uncertain target acceleration.*

**Theorem 6** *After  $s$  reaches the domain  $|s| \leq \mu$ , it may leave it. But, it is always guaranteed that a real sliding mode domain  $S$  exists as*

$$S = \{s, \dot{s} : |s| \leq \theta_1, |\dot{s}| \leq \theta_2, \theta_1 > \mu\}$$

where  $\theta_1$  and  $\theta_2$  are the boundaries,  $\theta_1 > 0$ ,  $\theta_2 > 0$ .

*Proof:* As the sliding surface enters the domain  $|s| \leq \mu$ ,  $\alpha$  satisfies (6.34). If  $\alpha \leq \alpha_m$ , the value of  $\alpha$  would start increasing such that  $\alpha = \alpha_m + \zeta \cdot t$ .  $|s|$  may become greater than  $\mu$  with decrease in gains  $\alpha$  and  $\beta$ . Then, the condition in Theorem 5 would be satisfied such that  $s$  reaches  $|s| \leq \mu$  again in finite time. Thus, it is assured that  $s$  always remains in the larger domain  $|s| \leq \theta_1$ ,  $\theta_1 > \mu$  [74].

If  $|s| \leq \mu$ ,  $|\dot{s}|$  satisfies the following inequality according to (6.8) and (6.9)

$$|\dot{s}| \leq (\alpha(T_1) + \delta)\mu^{1/2} + [\epsilon\alpha(T_1) + \gamma](T_2 - T_1) = \bar{\theta}_2$$

where  $T_1$  and  $T_2$  denote the time instants when  $s$  enters the domain  $|s| \leq \mu$  and leaves it, respectively.

When  $|s|$  satisfies  $\mu < |s| \leq \theta_1$ , then

$$|\dot{s}| \leq (\theta_1^{1/2} + \epsilon)(\alpha(T_2) + w_1\sqrt{\theta_1\gamma_1/2})(T_3 - T_2) + \delta\theta_1^{1/2} + \gamma(T_3 - T_2) = \tilde{\theta}_2$$

where  $T_3$  represents the time instant when  $s$  enters the domain  $|s| \leq \mu$  thereafter. Then the sliding mode domain becomes

$$S = \{s, \dot{s} : |s| \leq \theta_1, |\dot{s}| \leq \theta_2, \theta_1 > \mu\}$$

where  $\theta_2 = \max(\bar{\theta}_2, \tilde{\theta}_2)$ . ■

From the above discussion, it is observed that  $\alpha(t)$  and  $\beta(t)$  remain bounded. They also remain bounded in the presence of uncertainty which has been taken in the system (6.5) (i.e.  $a_{T\theta}$ ). The following propositions can be given:

*Proposition 1:* The adaptive gains  $\alpha(t)$  and  $\beta(t)$  remain bounded.

*Proof:* For the region  $\mu < |s| \leq \theta_1$ , a solution of (6.10) is

$$\alpha = \alpha(0) + w_1\sqrt{\gamma_1/2} \cdot t, \quad 0 \leq t \leq t_f \quad (6.36)$$

where  $t_f$  is the finite reaching time. Thus,  $\alpha(t)$  is bounded which results in bounded  $\beta(t)$ .

*Proposition 2:* In view of (6.24), the AST guidance law (6.10) forces the sliding variable  $s$  and its derivative to zero in finite time which can be expressed as:

$$t_f \leq \frac{2V^{1/2}(0)}{\zeta_0} \quad (6.37)$$

where  $\zeta_0 = \min(\tau, w_1, w_2)$ .

*Proof:* From (6.24), it is satisfied in finite time, given that its right-hand side is bounded and the adaptive gain  $\alpha(t)$  is increasing linearly with time in connection with (6.10). The inequality (6.37) is obtained by directly integrating inequality (6.32) taking into consideration that  $\Xi = 0$  due to the adaptive law (6.10).

### 6.3.2 Guidance Law with Extended State Observer

The ESO treats the uncertainties in the system as the extended state to be estimated. The estimated information of the uncertain target acceleration  $a_{T\theta}$  will be used in the guidance law (6.9). Considering  $a_{T\theta}$  as an extended state, the subsystem (6.5) becomes

$$\begin{aligned}\dot{x}_2 &= -\frac{2\dot{r}}{r}x_2 + \frac{a_{T\theta}}{r} - \frac{u}{r} \\ \dot{a}_{T\theta} &= d(t)\end{aligned}\tag{6.38}$$

where  $d(t)$  is the uncertain function expressed as the derivative of the target acceleration  $a_{T\theta}$ . So, the second-order dynamics of the ESO for the subsystem (6.5) can be used [120]

$$\begin{aligned}\dot{\hat{x}}_2 &= \frac{\hat{a}_{T\theta}}{r} - \beta_1 e_1 - \frac{2\dot{r}}{r}x_2 - \frac{u}{r} \\ \dot{\hat{a}}_{T\theta} &= \begin{cases} -\beta_2 |e_1|^{\alpha_1} \text{sign}(e_1), & |e_1| > \kappa \\ -\beta_2 e_1 / \kappa^{1-\alpha_1}, & \text{otherwise} \end{cases}\end{aligned}\tag{6.39}$$

where  $e_1 = \hat{x}_2 - x_2$  is the estimation error,  $\hat{x}_2$  and  $\hat{a}_{T\theta}$  represent the observer outputs and  $\beta_1, \beta_2$  denote the observer gains.

By choosing suitable values of the parameters  $\beta_1, \beta_2, \alpha_1$  and  $\kappa$ , one output of the ESO  $\hat{x}_2$  approaches to the actual  $x_2$  and another output  $\hat{a}_{T\theta}$  approaches to the uncertain quantity  $a_{T\theta}$ . With the estimated information of the target acceleration  $a_{T\theta}$  provided by the ESO, the guidance law (6.9) is improved as

$$\begin{aligned}u_{ESO} &= crx_2 - 2\dot{r}x_2 + r\alpha|s|^{1/2}\text{sign}(s) - rv + \hat{a}_{T\theta} \\ \dot{v} &= -\frac{\beta}{2}\text{sign}(s).\end{aligned}\tag{6.40}$$

Now, the obtained result can be described as:

**Theorem 7** *For the system (6.5) with the extended state observer (6.39), there exist observer gains  $\beta_1, \beta_2, \alpha_1$  and  $\kappa$  such that the estimated states  $\hat{x}_2$  and  $\hat{a}_{T\theta}$  converge to the actual states  $x_2$  and  $a_{T\theta}$ , respectively. So, the states of the system (6.5) can be maintained in a neighbourhood of the sliding surface (6.8) within finite time with the designed guidance law (6.40).*

*Proof:* The above theorem is proved in two parts:

*Part A:* In this part, we prove that the estimated states  $\hat{x}_2$  and  $\hat{a}_{T\theta}$  converge into the actual states  $x_2$  and  $a_{T\theta}$ , respectively.

Let us consider  $e_2 = \hat{a}_{T\theta} - a_{T\theta}$ . Taking the time derivative of  $e_1$  and  $e_2$ , one can obtain

$$\begin{aligned} \dot{e}_1 &= \frac{e_2}{r} - \beta_1 e_1 \\ \dot{e}_2 &= \begin{cases} -d(t) - \beta_2 |e_1|^{\alpha_1} \text{sign}(e_1), & |e_1| > \kappa \\ -d(t) - \beta_2 e_1 / \kappa^{1-\alpha_1}, & \text{otherwise.} \end{cases} \end{aligned} \quad (6.41)$$

The stability of the ESO is obtained by choosing suitable parameters  $\beta_1$  and  $\beta_2$ . As the observer dynamics is stable, the derivative of the error vector,  $\dot{e} = [\dot{e}_1, \dot{e}_2]^T = 0$ .

From (6.39), if  $|e_1| > \kappa$ , the estimation errors are

$$\begin{cases} |e_1| = |d(t)/\beta_2|^{1/\alpha_1} \\ |e_2| = r\beta_1 |d(t)/\beta_2|^{1/\alpha_1} \end{cases} \quad (6.42)$$

and if  $|e_1| \leq \kappa$ , the estimation errors are

$$\begin{cases} |e_1| = |d(t)\kappa^{1-\alpha_1}/\beta_2| \\ |e_2| = r\beta_1 |d(t)\kappa^{1-\alpha_1}/\beta_2| \end{cases} \quad (6.43)$$

With the help of (6.42) and (6.43), it is noted that the estimation error terms  $e_1$  and  $e_2$  are characterized by the values of  $\beta_1$ ,  $\beta_2$ ,  $\alpha_1$ , and  $\kappa$ . By appropriately tuning the parameters, the estimation errors can be forced towards the origin, such that the system state  $x_2$  and extended state  $a_{T\theta}$  are effectively estimated. The parameters can be chosen as  $\beta_1 > 0$ ,  $\beta_2 > 0$ ,  $0 < \alpha_1 < 1$  and  $0 < \kappa < 1$ .

*Part B:* Substituting guidance law (6.40) into (6.11) yields

$$\begin{aligned} \dot{s} &= -\alpha |s|^{1/2} \text{sign}(s) + w_* + \psi(x) \\ \dot{w}_* &= -\frac{\beta}{2} \text{sign}(s) + \dot{g}^*(t) \end{aligned} \quad (6.44)$$

where  $w_* = v + g^*(t)$ ,  $\psi(x, t) = -cx_2 + k\eta|x_1|^{\eta-1}x_2$  and  $g^*(t) = \frac{a_{T\theta} - \hat{a}_{T\theta}}{r}$ .

Let the derivative of  $g^*(t)$  remains bounded satisfying the inequality

$$|\dot{g}^*(t)| \leq \gamma^* \quad (6.45)$$

where  $\gamma^*$  is bounded constant.

In view of (6.45), it can be obtained that

$$\dot{g}^*(t) = \frac{\Delta_2^*(x, t)}{2} \text{sign}(s) = \frac{\Delta_2^*(x, t)}{2} \frac{z_1}{|z_1|} \quad (6.46)$$

where  $\Delta_2^*(x, t)$  is the bounded as

$$0 < \Delta_2(x, t) \leq 2\gamma^*.$$

From (6.14), the system (6.18) can be modified as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = A^*(z_1) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (6.47)$$

where

$$A^*(z_1) = \frac{1}{2|z_1|} \begin{bmatrix} -(\alpha - \Delta_1(x, t)) & 1 \\ -(\beta - \Delta_2^*(x, t)) & 0 \end{bmatrix}$$

The remaining part of the proof is similar to *Theorems 5* and *6*, and hence is omitted here. ■

### 6.3.3 Simulation results

The performance of the designed AST guidance law is illustrated through the simulation results. The initial position co-ordinates of the missile are  $x_{M0} = 0$  m and  $y_{M0} = 0$  m. The initial velocity of the missile is  $V_M = 300$  m/s. The initial value of flight-path angle is taken as  $\phi_M = 60^\circ$ . The initial position co-ordinates of the target are  $x_{T0} = 5000$  m and  $y_{T0} = 5000$  m. Initial velocity of the target is  $V_T = 200$  m/s. Its initial flight-path angle is  $\phi_T = 120^\circ$ . The design parameters of the AST guidance law are selected as  $\epsilon = 0.5$ ,  $\gamma_1 = 100$ ,  $w_1 = 500$ ,  $\eta = 0.8$ ,  $k = 10$ ,  $c = 2.5$ ,  $\mu = 0.9$ ,  $\alpha_1 = 0.09$ ,  $\kappa = 1 \times 10^{-4}$ ,  $\alpha_m = \zeta = 0.006$ ,  $\beta_1 = 10$ , and  $\beta_2 = 400$ . The simulations are performed for three cases.

#### ***Case 1: Non-maneuvering target***

For the non-maneuvering target, its acceleration  $a_{T\theta} = 0$  m/s<sup>2</sup>. By using the proposed guidance law (6.40), the LOS rate  $x_2$  and the missile acceleration  $u_{ESO}$  converge to the origin in finite time. Also, the estimated state  $\hat{a}_{T\theta}$  tracks to the target acceleration  $a_{T\theta}$  and shown in Figures 6.2 (a) and 6.2 (b).

#### ***Case 2: Constant maneuvering target***

Considering constant maneuvering, the target acceleration is taken as  $a_{T\theta} = 100$  m/s<sup>2</sup>. In view of the proposed guidance law (6.40), the LOS rate  $x_2$  converges to the origin and the missile acceleration  $u_{ESO}$  and the estimated target acceleration  $\hat{a}_{T\theta}$  tracks to the target acceleration  $a_{T\theta}$  in finite time. it means the target acceleration can be estimated by using

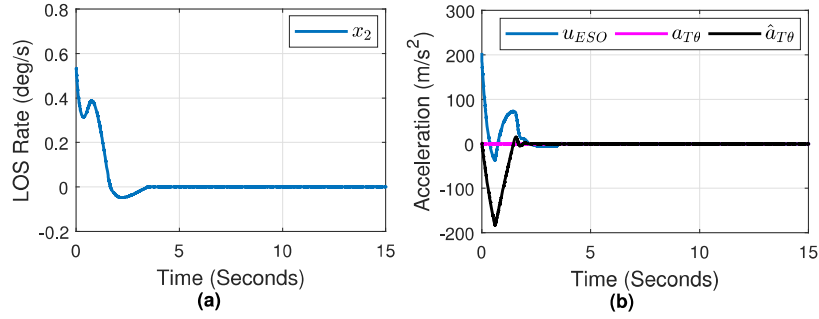


Figure 6.2: For the non-maneuvering target, the LOS rate  $x_2$  and the accelerations are shown in Figure 6.2 (a) and Figure 6.2 (b), respectively.

the ESO (6.39), if the target acceleration is not known to us and illustrated in Figure 6.3 (a) and Figure 6.3 (b).

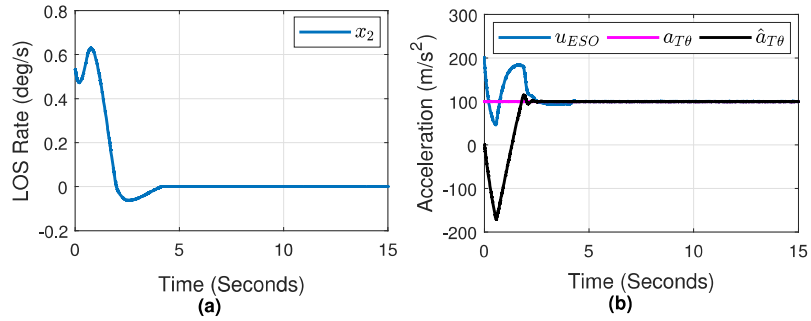


Figure 6.3: For the constant maneuvering target, the evolution of the LOS rate  $x_2$  and the accelerations are illustrated in Figure 6.3 (a) and Figure 6.3 (b), respectively.

### ***Case 3: Time-varying maneuvering target***

In this case, the target acceleration is taken as  $a_{T\theta} = 100\sin(1.5t)$ . In accordance with the proposed guidance law (6.40), The simulation results in Figure 6.4 (a) and Figure 6.4 (b) show the LOS rate converges to the origin as well as the observer state  $\hat{a}_{T\theta}$  estimating the target acceleration  $a_{T\theta}$  in finite time.

From the above discussion and simulation results, it is clear that the LOS rate is converging in the desired manner. With the designed ESO, the unknown target acceleration is estimated in finite time which is further utilized in designing the AST guidance law. The characteristic smoothness of the STA reflects in the simulated response of the guidance laws. By replacing the estimated state  $\hat{a}_{T\theta}$ , the proposed guidance law (6.40) is also used for the case when the target acceleration is known to us. Satisfactory performance of the designed law is evident from the obtained simulation results.

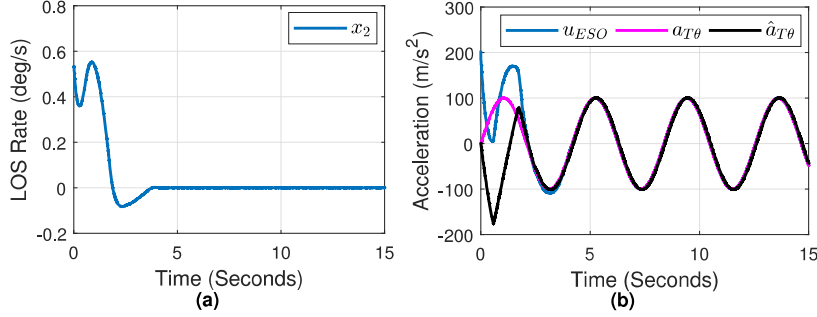


Figure 6.4: For the time-varying maneuvering target, the LOS rate  $x_2$  and the accelerations are depicted in Figure 6.4 (a) and Figure 6.4 (b), respectively.

## 6.4 Adaptive Super-Twisting Guidance Law: An Event-Triggered Approach

Let us consider nonlinear dynamical system

$$\dot{x} = f(x) + g(x)u + bd; \quad x(t_0) = x_0 \quad (6.48)$$

where  $b = [0, 0, \dots, 1]^T \in \mathbb{R}^n$ ,  $x \in \mathbb{R}^n$  denotes state of the system,  $u \in \mathbb{R}$  represents the control signal.  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are the nonlinear function.  $t_0 \in \mathbb{R}_{\geq 0}$  is the initial time. The disturbance  $d \in \mathbb{R}$  is continuous differentiable unknown function with known bound such that

$$|d| \leq d_1, \quad |\dot{d}| \leq d_1^*, \quad d_1, d_1^* > 0. \quad (6.49)$$

Assume that the sequence of triggering instants are  $t_k, \forall k \in \mathbb{Z}_{\geq 0}$ , and discrete error between two successive intervals is  $e(t) = x(t_k) - x(t)$  for  $t \in [t_k, t_{k+1})$ .

**Assumption 12** *The function  $f$  in (6.48) holds the Lipschitz property in a compact set  $D \subset \mathbb{R}^n$  such that*

$$\|f(z_1) - f(z_2)\| \leq L\|z_1 - z_2\| \quad (6.50)$$

where  $L$  is the positive Lipschitz constant.

**Assumption 13** *A sliding manifold  $s(x, t) \in \mathbb{R}$  is chosen such that the system (6.48) is satisfied on the sliding manifold  $s(x, t) = 0$ .*

Suppose that the relative degree of the system (6.48) is one for the sliding manifold  $s(x, t)$  in relation to the control signal  $u$ . The system (6.48) has a stable internal dynamics. Therefore, the dynamics of sliding manifold is

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}f(x) + \frac{\partial s}{\partial x}g(z)u + \frac{\partial s}{\partial x}bd \\ &= \psi(x, t) + D + h(x, t)u\end{aligned}\quad (6.51)$$

where  $\psi(x, t) := \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}f(x)$ ,  $h(x, t) := \frac{\partial s}{\partial x}g(x)$  and  $D := \frac{\partial s}{\partial x}bd$ .

**Assumption 14** *The function  $\psi(x, t) \in \mathbb{R}$  is bounded and  $h(x, t) \in \mathbb{R}$  is known with  $h(x, t) \neq 0 \forall x$  and  $t \in [0, \infty)$ .*

### 6.4.1 Design of Guidance Law

The key objective of this part is to construct an adaptive super-twisting guidance law based on event-triggered for the planar missile-target engagement geometry which is presented in Figure 6.1.

Recall the previous equation which has been mentioned in (6.5) as follows

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{2\dot{r}}{r}x_2 + \frac{a_{T\theta}}{r} - \frac{u}{r}.\end{aligned}\quad (6.52)$$

The sliding surface for the given system (6.52) is designed as

$$s = x_2 + k|x_1|^\eta \text{sign}(x_1) \quad (6.53)$$

where  $k$  is a positive constant and  $0.5 < \eta < 1$ .

For the system (6.52), an event-triggered adaptive super-twisting algorithm (ETASTA) based guidance law is defined as

$$u = r(t_k)F(t_k) + r(t_k)\alpha(t_k)|s(t)|^{1/2}\text{sign}(s(t_k)) + \int_{t_k}^t r(t_k)\beta(t_k)\text{sign}(s(t_k))dt \quad (6.54)$$

where the function  $F(t_k) := -2\dot{r}(t_k)x_2(t_k)/r(t_k) + F_1(x_1(t_k))x_2(t_k)$  in which  $F_1(x_1(t_k)) = \eta k|x_1(t_k)|^{\eta-1}$ . To avoid the singularity,  $F_1(x_1(t_k)) = 0$  if  $x_1(t_k) = 0$  and  $s(t_k) \neq 0$ , otherwise,  $F_1(x_1(t_k)) = \eta k|x_1(t_k)|^{\eta-1}$ .

During the course of the guidance process, we have  $0 < r(t) < r(0)$ ,  $\dot{r}(t) < 0$ ,  $\forall t > 0$ .

The adaptive gains  $\alpha$  and  $\beta$  are updated by the adaptive laws

$$\dot{\alpha} = w_1\text{sign}(|s| - \mu), \quad \beta = \xi\alpha \quad (6.55)$$

where  $w_1$ ,  $\mu$  and  $\xi$  are arbitrary positive constants.

The triggering rule is designed as

$$t_{k+1} = \inf\{t : t > t_k, L\|e(t)\| \geq \theta(|s(t_k)|^{1/2} + \Xi)\} \quad (6.56)$$

where  $0 < \theta < 1$  and  $\Xi, L > 0$ .

Therefore, the triggering condition for stability is defined as

$$L\|e(t)\| < \theta(|s(t_k)|^{1/2} + \Xi), \forall t \geq 0. \quad (6.57)$$

**Theorem 8** *Suppose that the assumptions 12, 13 and 14 holds for the system (6.52). If there exist the guidance law (6.54), adaptive laws (6.55) and triggering rule (6.56) such that adaptive gain*

$$\alpha \geq \alpha(t_k) > \frac{\xi(4d_1^* + 1) + \gamma + 4\xi^2}{\gamma} + \frac{(\gamma + 4\xi^2 + 2d_1^*)^2}{4\gamma\xi}.$$

where  $\gamma$  is a positive constant. Then, the finite-time convergence of the LOS rate is achieved if the sliding manifold satisfies the following inequality

$$\left\{ z \in \mathbb{R}^2 : |s| \leq \max\left(\frac{\rho^2}{(\lambda_{\min}(\tilde{Q})c_0)^2}, \frac{\theta(|s(t_k)|^{1/2} + \Xi)}{L}\right) \right\} \quad (6.58)$$

where  $0 < c_0 < 1$  and  $\rho$  is a positive constant.

*Proof:* At the initial time, we consider states and sliding manifold  $s$  to be non-zero. The time derivative of (6.53) is

$$\dot{s} = \dot{z}_2 + \eta k |z_1|^{\eta-1} z_2. \quad (6.59)$$

Invoking (6.52) and (6.54), one can rewrite (6.59) as

$$\dot{s}(t) = F(t) - F(t_k) - \alpha(t_k)|s(t_k)|^{1/2}\text{sign}(s(t_k)) - \int_{t_k}^t \beta(t_k)\text{sign}(s(t_k))dt + \frac{a_{T\theta}}{r}. \quad (6.60)$$

Eq. (6.60) can also be written as

$$\dot{s} = F(t) - F(t_k) - \alpha(t_k)|s(t_k)|^{1/2}\text{sign}(s(t_k)) - \int_{t_k}^t \beta(t_k)\text{sign}(s(t_k))dt + d(t) \quad (6.61)$$

where  $d = a_{T\theta}/r$ .

Further, (6.61) can also be represented as

$$\begin{aligned} \dot{s} &= F(t) - F(t_k) - \alpha(t_k)|s(t_k)|^{1/2}\text{sign}(s(t_k)) + \nu \\ \dot{\nu} &= -\beta(t_k)\text{sign}(s(t_k)) + \dot{d}(t) \end{aligned} \quad (6.62)$$

where  $\nu = -\int_{t_k}^t \beta(t_k)\text{sign}(s(t_k))dt + d(t)$ .

Let us define the new state variables as

$$\zeta(t) \triangleq [\zeta_1(t) \ \zeta_2(t)]^T = [|s(t)|^{1/2}\text{sign}(s(t)) \ \nu(t)]^T. \quad (6.63)$$

The equivalent representation of system (6.63) becomes

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \frac{1}{2|\zeta_1(t)|} \left[ A \begin{bmatrix} \zeta_1(t_k) \\ \zeta_2(t) \end{bmatrix} + B \right] \quad (6.64)$$

where

$$A = \begin{bmatrix} -\alpha(t_k) & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} F(t) - F(t_k) \\ 2|\zeta_1(t)| \left( -\beta(t_k)\text{sign}(s(t_k)) + \dot{d} \right) \end{bmatrix}.$$

Let us consider the Lyapunov function as

$$V(\zeta, \alpha, \beta) = V_0(\zeta) + \frac{1}{2}(\alpha - \bar{\alpha})^2 + \frac{1}{2}(\beta - \bar{\beta})^2 \quad (6.65)$$

where  $\bar{\alpha} > 0$  and  $\bar{\beta} > 0$ , i.e.,  $|\alpha| \leq \bar{\alpha}$ ,  $|\beta| \leq \bar{\beta}$ , are positive constants [136] and

$$V_0(\zeta) = \zeta^T(t)P\zeta(t), \quad P = \begin{bmatrix} \gamma + 4\xi^2 & -2\xi \\ -2\xi & 1 \end{bmatrix} \quad \text{with } \gamma, \xi > 0. \quad (6.66)$$

The time derivative of (6.65) yields

$$\dot{V}(\zeta, \alpha, \beta) = \dot{V}_0(\zeta) + (\alpha - \bar{\alpha})\dot{\alpha} + (\beta - \bar{\beta})\dot{\beta}. \quad (6.67)$$

In accordance of (6.64),  $\dot{V}_0(\zeta)$  can be written as

$$\dot{V}_0(\zeta) = \frac{1}{|\zeta_1(t)|} \left\{ \begin{bmatrix} \zeta_1(t_k) & \zeta_2(t) \end{bmatrix} A^T P \zeta(t) + \zeta^T(t) P B \right\}. \quad (6.68)$$

One can rewrite (6.68) as

$$\begin{aligned} \dot{V}_0(\zeta) = \frac{1}{|\zeta_1(t)|} \left\{ \right. & -(\gamma + 4\xi^2)\alpha(t_k)\zeta_1(t_k)\zeta_1(t) + (\gamma + 4\xi^2)\zeta_1(t)\zeta_2(t) + 2\alpha(t_k)\xi\zeta_1(t_k)\zeta_2(t) \\ & - 2\xi\zeta_2^2(t) + (\gamma + 4\xi^2)(F(t) - F(t_k))\zeta_1(t) - 2\xi(F(t) - F(t_k))\zeta_2(t) \\ & + 4\xi\beta(t_k)|\zeta_1(t)|\zeta_1(t)\text{sign}(s(t_k)) - 2\beta(t_k)|\zeta_1(t)|\zeta_2(t)\text{sign}(s(t_k)) \\ & \left. - 4\xi\zeta_1(t)|\zeta_1(t)|\dot{d} + 2|\zeta_1(t)|\zeta_2(t)\dot{d} \right\}. \quad (6.69) \end{aligned}$$

From the previous assumption  $|d| \leq d_1^*$ , (6.69) can be represented as

$$\begin{aligned} \dot{V}_0(\zeta) \leq \frac{1}{|\zeta_1(t)|} \left\{ & -(\gamma + 4\xi^2)\alpha(t_k)\zeta_1(t_k)\zeta_1(t) + (\gamma + 4\xi^2)\zeta_1(t)\zeta_2(t) + 2\alpha(t_k)\xi\zeta_1(t_k)\zeta_2(t) \right. \\ & - 2\xi\zeta_2^2(t) + (\gamma + 4\xi^2)(F(t) - F(t_k))\zeta_1(t) - 2\xi(F(t) - F(t_k))\zeta_2(t) \\ & + 4\xi\beta(t_k)\zeta_1(t)|\zeta_1(t)|\text{sign}(s(t_k)) - 2\beta(t_k)\zeta_2(t)|\zeta_1(t)|\text{sign}(s(t_k)) \\ & \left. + 4\xi d_1^*\zeta_1(t)|\zeta_1(t)| + 2d_1^*|\zeta_1(t)|\zeta_2(t) \right\}. \end{aligned} \quad (6.70)$$

For the above discussions, we have considered two different cases. The case-1  $\text{sign}(s(t_k)) = \text{sign}(s(t))$  and the case-2  $\text{sign}(s(t_k)) \neq \text{sign}(s(t))$ .

**Case-1:** According to (6.55), when  $\text{sign}(s(t_k)) = \text{sign}(s(t))$ ,  $|s(t_k)| > \mu$  and  $\dot{\alpha} > 0$ , the sliding trajectory doesn't reach the sliding surface.

From the previous assumption  $\zeta_1(t) = |s|^{1/2}\text{sign}(s(t)) \implies |\zeta_1(t)| = |s|^{1/2}$ , one can write (6.70) as

$$\begin{aligned} \dot{V}_0(\zeta) \leq \frac{1}{|\zeta_1(t)|} \left\{ & -(\gamma + 4\xi^2)\alpha(t_k)|\zeta_1(t_k)||\zeta_1(t)| + (\gamma + 4\xi^2)|\zeta_1(t)||\zeta_2(t)| \right. \\ & + 2\xi\alpha(t_k)|\zeta_1(t_k)||\zeta_2(t)| - 2\xi\zeta_2^2(t) + (\gamma + 4\xi^2)(F(t) - F(t_k))\zeta_1(t) - 2\xi(F(t) \\ & - F(t_k))\zeta_2(t) + 4\xi\beta(t_k)\zeta_1^2(t) - 2\beta(t_k)|\zeta_1(t)||\zeta_2(t)| + 4\xi d_1^*|\zeta_1(t)||\zeta_1(t)| \\ & \left. + 2d_1^*|\zeta_1(t)||\zeta_2(t)| \right\}. \end{aligned} \quad (6.71)$$

From (6.50), i.e.,  $|F(t) - F(t_k)| \leq L||z(t) - z(t_k)|| \leq L||e(t)||$ , (6.71) yields

$$\begin{aligned} \dot{V}_0(\zeta) \leq \frac{1}{|\zeta_1(t)|} \left\{ & 4\xi(\beta(t_k) + d_1^*)\zeta_1^2(t) - 2\xi\zeta_2^2(t) + (\gamma + 4\xi^2 - 2\beta(t_k) + 2d_1^*)|\zeta_1(t)||\zeta_2(t)| \right. \\ & - (\gamma + 4\xi^2)\alpha(t_k)|\zeta_1(t_k)||\zeta_1(t)| + 2\xi\alpha(t_k)|\zeta_1(t_k)||\zeta_2(t)| \\ & \left. + (\gamma + 4\xi^2)|\zeta_1(t)|L||e(t)|| + 2\xi|\zeta_2(t)|L||e(t)|| \right\}. \end{aligned} \quad (6.72)$$

From the fact  $|s(t_k)| = |s(t) + ||z(t) - z(t_k)|||$ , one can write as

$$|s(t) - (||e(t)||)| \leq |s(t_k)| \leq |s(t) + (||e(t)||)|. \quad (6.73)$$

Using (6.57), we can write (6.73) as

$$|s(t)| - \frac{|s(t_k)|^{1/2} + \Xi}{L} \leq |s(t_k)| \leq |s(t)| + \frac{|s(t_k)|^{1/2} + \Xi}{L}. \quad (6.74)$$

The right hand of inequality (6.74) can be calculated as

$$\left( |s(t_k)|^{1/2} - \frac{1}{2L} \right)^2 \leq |s(t)| + \frac{1}{4L^2} + \frac{\Xi}{L}. \quad (6.75)$$

After some mathematical computation, one can obtain

$$\left| |s(t_k)| - \frac{1}{2L} \right| \leq \left( |s(t)| + \frac{1}{4L^2} + \frac{\Xi}{L} \right)^{1/2}. \quad (6.76)$$

Invoking the inequality  $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$  with  $a > 0$  and  $b > 0$ , (6.76) yields

$$|s(t_k)|^{1/2} \leq |s(t)|^{1/2} + \frac{1}{2L} + \left( \frac{1}{4L^2} + \frac{\Xi}{L} \right) = |s(t)|^{1/2} + \epsilon_1. \quad (6.77)$$

Furthermore, the left hand side of inequality (6.73) can be written as

$$|s(t)| \leq |s(t)| + \frac{1}{4L^2} \leq \left( |s(t_k)|^{1/2} + \frac{1}{2L} \right)^2 + \frac{\Xi}{L}. \quad (6.78)$$

Using the inequality  $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$  with  $a > 0$  and  $b > 0$ , the (6.78) can be represented as

$$|s(t)|^{1/2} - \left( \frac{1}{2L} + \left( \frac{\Xi}{L} \right)^{1/2} \right) = |s(t)|^{1/2} - \epsilon_2 \leq |s(t_k)|^{1/2}. \quad (6.79)$$

In view of (6.77) and (6.79), we have

$$|s(t)|^{1/2} - \epsilon_2 \leq |s(t_k)|^{1/2} \leq |s(t)|^{1/2} + \epsilon_1. \quad (6.80)$$

Invoking (6.63), one can written as

$$|\zeta_1(t)| - \epsilon_2 \leq |\zeta_1(t_k)| \leq |\zeta_1(t)| + \epsilon_1. \quad (6.81)$$

Substitution of (6.80) into (6.72), yields

$$\begin{aligned} \dot{V}_0(\zeta) \leq & \frac{1}{|\zeta_1(t)|} \left\{ 4\xi(\beta(t_k) + d_1^*)\zeta_1^2(t) - 2\xi\zeta_2^2(t)(t) + (\gamma + 4\xi^2 - 2\beta(t_k) + 2d_1^*)|\zeta_1(t)||\zeta_2(t)| \right. \\ & - (\gamma + 4\xi^2)\alpha(t_k)|\zeta_1(t)|(|\zeta_1(t)| - \epsilon_2) + 2\xi\alpha(t_k)|\zeta_2(t)| \\ & \left. \times (|\zeta_1(t)| + \epsilon_1) + (\gamma + 4\xi^2)|\zeta_1(t)|(|\zeta_1(t_k)| + \Xi) + 2\xi|\zeta_2(t)|(|\zeta_1(t_k)| + \Xi) \right\}. \quad (6.82) \end{aligned}$$

Simplified form of (6.82) can be expressed as

$$\begin{aligned} \dot{V}_0(\zeta) \leq & \frac{1}{|\zeta_1(t)|} \left\{ 4\xi(\beta(t_k) + d_1^*)\zeta_1^2(t) - 2\xi\zeta_2^2(t) + (\gamma + 4\xi^2 - 2\beta(t_k) + 2d_1^*)|\zeta_1(t)||\zeta_2(t)| \right. \\ & - (\gamma + 4\xi^2)\alpha(t_k)|\zeta_1(t)|(|\zeta_1(t)| - \epsilon_2) + 2\xi\alpha(t_k)|\zeta_2(t)|(|\zeta_1(t)| + \epsilon_1) + (\gamma + 4\xi^2)|\zeta_1(t)| \\ & \left. \times (|\zeta_1(t)| + \epsilon_1 + \Xi) + 2\xi|\zeta_2(t)|(|\zeta_1(t)| + \epsilon_1 + \Xi) \right\}. \quad (6.83) \end{aligned}$$

Furthermore, inequality (6.83) can be written as

$$\begin{aligned} \dot{V}_0(\zeta) \leq & \frac{1}{|\zeta_1(t)|} \left\{ [4\xi(\beta(t_k) + d_1^*) + (\gamma + 4\xi^2)(1 - \alpha(t_k))] \times \zeta_1^2(t) \right. \\ & \left. + [\gamma + 4\xi^2 - 2\beta(t_k) + 2d_1^* + 2\xi\alpha(t_k)] |\zeta_1(t)| |\zeta_2(t)| - 2\xi\zeta_2^2(t) + W \right\} \end{aligned} \quad (6.84)$$

$$\begin{aligned} \text{where } W = & [(\gamma + 4\xi^2)(\epsilon_2\alpha(t_k) + \epsilon_1 + \Xi)] |\zeta_1(t)| + 2\xi(\alpha(t_k)\epsilon_1 + \epsilon_1 + \Xi) |\zeta_2(t)| \\ \leq & [(\gamma + 4\xi^2)(\epsilon_2\bar{\alpha} + \epsilon_1 + \Xi) + 2\xi(\bar{\alpha}\epsilon_1 + \epsilon_1 + \Xi)] \|\zeta(t)\| = \rho \|\zeta(t)\| \end{aligned} \quad (6.85)$$

with  $\rho = (\gamma + 4\xi^2)(\epsilon_2\bar{\alpha} + \epsilon_1 + \Xi) + 2\xi(\bar{\alpha}\epsilon_1 + \epsilon_1 + \Xi)$ .

It can be shown from the (6.85) that the  $\rho > 0$  is bounded. Considering (6.85), the (6.84) can be expressed as

$$\begin{aligned} \dot{V}_0(\zeta) \leq & -\frac{1}{|\zeta_1(t)|} \left\{ \begin{bmatrix} \zeta_1(t) & \zeta_2(t) \end{bmatrix} \tilde{Q} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + W \right\} \\ \leq & -\frac{1}{|\zeta_1(t)|} \tilde{\zeta}^T(t) \tilde{Q} \tilde{\zeta}(t) + \frac{\rho}{|\zeta_1(t)|} \|\zeta(t)\| \end{aligned} \quad (6.86)$$

where  $\tilde{\zeta}(t) = [\zeta_1(t) \ \zeta_2(t)]^T$  and

$$\tilde{Q} = \begin{bmatrix} -4\xi(\beta(t_k) + d_1^*) - (\gamma + 4\xi^2)(1 - \alpha(t_k)) & \star \\ -\frac{1}{2}(\gamma + 4\xi^2 - 2\beta(t_k) + 2d_1^* + 2\xi\alpha(t_k)) & 2\xi \end{bmatrix}. \quad (6.87)$$

For the positive definiteness of the matrix  $\tilde{Q}$ , let us take  $\beta = \xi\alpha$ . A minimal eigenvalue of the matrix  $\tilde{Q}$  is defined as  $\lambda_{\min}(\tilde{Q}) \geq \xi$  if

$$\alpha \geq \alpha(t_k) > \frac{\xi(4d_1^* + 1) + \gamma + 4\xi^2}{\gamma} + \frac{(\gamma + 4\xi^2 + 2d_1^*)^2}{4\gamma\xi}. \quad (6.88)$$

The condition of (6.88) ensures in finite-time.

By using  $\|\tilde{\zeta}\| = \|\zeta\|$ , the following inequality holds  $\lambda_{\min}(\tilde{Q})\|\zeta\|^2 \leq \tilde{\zeta}^T \tilde{Q} \tilde{\zeta} \leq \lambda_{\max}(\tilde{Q})\|\zeta\|^2$ , one can write (6.86) as

$$\dot{V}_0(\zeta) \leq -\frac{1}{|\zeta_1(t)|} \lambda_{\min}(\tilde{Q}) \|\zeta(t)\|^2 + \frac{\rho}{|\zeta_1(t)|} \|\zeta(t)\|. \quad (6.89)$$

With the help of (6.89), we can write (6.67) as

$$\begin{aligned} \dot{V}(\zeta, \alpha, \beta) \leq & -\frac{1}{|\zeta_1(t)|} \lambda_{\min}(\tilde{Q}) \|\zeta(t)\|^2 + \frac{\rho}{|\zeta_1(t)|} \|\zeta(t)\| + (\alpha - \bar{\alpha})\dot{\alpha} + (\beta - \bar{\beta})\dot{\beta} \\ = & -\frac{\lambda_{\min}(\tilde{Q})}{|\zeta_1(t)|} \|\zeta(t)\|^2 + \rho \frac{\|\zeta(t)\|}{|\zeta_1(t)|} - |\alpha - \bar{\alpha}|w_1 - |\beta - \bar{\beta}|\xi w_1 \\ \leq & -\frac{\lambda_{\min}(\tilde{Q})}{|\zeta_1(t)|} (1 - c_0) \|\zeta(t)\|^2 + \frac{\rho}{|\zeta_1(t)|} \|\zeta(t)\| - |\alpha - \bar{\alpha}|w_1 \\ & - |\beta - \bar{\beta}|\xi w_1 - c_0 \frac{\lambda_{\min}(\tilde{Q})}{|\zeta_1(t)|} \|\zeta(t)\|^2, \quad 0 < c_0 < 1. \end{aligned} \quad (6.90)$$

Applying Rayleigh inequality and  $|\zeta_1(t)| \leq \|\zeta(t)\|$ , one gets

$$\dot{V}(\zeta, \alpha, \beta) \leq -r_0 V_0^{1/2} - |\alpha - \bar{\alpha}|w_1 - |\beta - \bar{\beta}|\xi w_1 + \frac{\rho}{|\zeta_1(t)|} \|\zeta(t)\| - \frac{1}{|\zeta_1(t)|} \lambda_{\min}(\tilde{Q})c_0 \|\zeta(t)\|^2 \quad (6.91)$$

where  $r_0 = \lambda_{\min}(\tilde{Q})(1 - c_0)/\lambda_{\min}^{1/2}(P)$ .

Using the inequality  $(a^2 + b^2 + c^2)^{1/2} \leq |a| + |b| + |c|$  satisfies  $a, b, c \in \mathbb{R}$ , one can write (6.91) as

$$\dot{V}(\zeta, \alpha, \beta) \leq -r_1 V_0^{1/2} + \rho \frac{\|\zeta(t)\|}{|\zeta_1(t)|} - c_0 \frac{\lambda_{\min}(\tilde{Q})}{|\zeta_1(t)|} \|\zeta(t)\|^2 \quad (6.92)$$

where  $r_1 = \min(r_0, \sqrt{2}w_1, \sqrt{2}\xi w_1)$ .

It can be derived from (6.92) that  $\dot{V} \leq -r_1 V^{1/2}$  if  $\rho < \lambda_{\min}(\tilde{Q})c_0 \|\zeta(t)\|$ . As a result, the system (6.52) guaranteed to be finite-time stable and the sliding surface converge to the region  $|s(t)| \leq \rho^2 / (\lambda_{\min}(\tilde{Q})c_0)^2$ .

**Case-2:** The relation (6.57) prevents the sliding trajectory from increasing beyond a specific amount when  $\text{sign}(s(t)) \neq \text{sign}(s(t_k))$ . When  $\text{sign}(s(t)) \neq \text{sign}(s(t_k))$ , it will be proved that the sliding trajectory is eventually confined.

The size of ultimate bound can be calculated by finding the maximum deviation of the sliding trajectory in any time interval  $t \in [t_k, t_{k+1})$  as  $|s(t_k) - s(t)| \leq \eta(|s(t_k)|^{1/2} + \Xi)/L$ . When  $s(t_k) = 0$ , the maximum value of sliding band can be found.

As soon as  $|s(t)|$  becomes greater than  $\mu$ , case-1 holds, such that  $|s(t)|$  reaches the domain  $|s(t)| \leq \rho^2 / (\lambda_{\min}(\tilde{Q})c_0)^2$  in finite-time. Then, due to relation (6.57), the sliding trajectory remains ultimate bounded within a bound in the case-2.

As a result of the study in the cases-1 and case-2, the sliding manifold eventually reaches the following region

$$\Omega = \left\{ |s| \leq \max \left( \frac{\rho^2}{(\lambda_{\min}(\tilde{Q})c_0)^2}, \frac{\theta(|s(t_k)|^{1/2} + \Xi)}{L} \right) \right\} \quad (6.93)$$

and remains for all future time. The proof is completed. ■

## 6.4.2 Analysis of Inter-Event Time

In this part, we show that the inter-execution times have a positive lower bound. The  $k^{\text{th}}$  inter-event time for this triggering sequence is defined as  $T_k = t_{k+1} - t_k, \forall k \in \mathbb{Z}_{\geq 0}$  based on the triggering circumstances.

**Theorem 9** For the given system (6.52), if the guidance law (6.54), the adaptive gains (6.55) and the triggering condition (6.56) are satisfied. Then, the inter-event time  $T_k$  must always be lower bound with a positive value.

*Proof:* Let  $\varphi = \{t \in [t_k, \infty) : \|e(t)\| = 0\}$ ,  $\forall t \in [t_k, \infty) \setminus \varphi$ . It can be found that

$$\begin{aligned} \frac{d}{dt}\|e(t)\| &\leq \left\| \frac{de(t)}{dt} \right\| = \left\| \frac{dz(t)}{dt} \right\| = \left\| \left( F(t) - F(t_k) \right. \right. \\ &\quad \left. \left. - \alpha(t_k)|s(t_k)|^{1/2}\text{sign}(s(t_k)) - \int_{t_k}^t \beta(t_k)\text{sign}(s(t_k))dt + \frac{a_{T\sigma}}{R} \right) \right\| \\ &\leq \|F(t) - F(t_k)\| + |\alpha(t_k)||s(t_k)|^{1/2} + |\beta(t_k)|(t - t_k) + d \\ &\leq L\|e(t)\| + K + |\beta(t_k)|T_k \end{aligned}$$

where  $K = |\alpha(t_k)||s(t_k)|^{1/2} + |d|$ .

After solving the above differential equation using the comparison principle, we get

$$\|e(t)\| \leq \frac{K + |\beta(t_k)|T_k}{L} (e^{L(t-t_k)} - 1), \quad t \in [t_k, t_{k+1}).$$

In the interval  $t \in [t_k, t_{k+1})$ , the error  $\|e(t)\|$  increase from 0 to  $\max\{\rho^2/(\lambda_{\min}(\tilde{Q})c_0)^2, \eta(|s(t_k)|^{1/2} + \Xi)/L\}$ . Therefore,  $\omega \leq \frac{K+|\beta(t_k)|T_k}{L}(e^{LT_k} - 1)$ , where  $\omega = \max\{\rho^2/(\lambda_{\min}(\tilde{Q})c_0)^2, \eta(|s(t_k)|^{1/2} + \Xi)/L\}$ . Consider

$$\Pi(T_k) = \frac{K + |\beta(t_k)|T_k}{L} (e^{LT_k} - 1) - \omega. \quad (6.94)$$

The function  $\Pi(T_k)$  is monotonically increasing  $\forall T_k \geq 0$  and  $\Pi(0) = -\omega < 0$ , the solution of (6.94) is  $T_k = \kappa > 0$ ,  $\Pi(\kappa) = 0$  (i.e.,  $T_k$  is a positive bound). Hence, the triggering sequence  $t_k$ ;  $k = 1, 2, \dots, N$  is admissible and Zeno-free phenomenon.  $\blacksquare$

### 6.4.3 Simulation results

The performance of an ETASTA guidance law is demonstrated via similar practical example, as noted in the section 6.3.3. The missile's starting location is considered to be  $x_{M0} = 0$  m and  $y_{M0} = 0$  m. The missile's starting velocity and path angle are  $V_M = 300$  m/s and  $\gamma_M = 60^\circ$ , respectively. The target's starting positions are set to  $x_{T0} = 5000$  m and  $y_{T0} = 5000$  m. The target's starting velocity and angle of flight are  $V_T = 200$  m/s and  $\gamma_T = 120^\circ$ , respectively.

The design parameters of the ETASTA guidance law are selected as  $\xi = 0.02$ ,  $w_1 = 0.002$ ,

$\eta = 0.6$ ,  $k = 1 \times 10^{-5}$ ,  $\mu = 6 \times 10^{-5}$ ,  $L = 50$ ,  $\theta = 0.8$ , and  $\Xi = 0.005$ . The simulations are performed for two cases.

**Case 1: Non-maneuvering target**

Assume the acceleration is  $a_{T\theta} = 0 \text{ m/s}^2$ . The LOS rate converges to zero or a small neighborhood of zero in finite-time using the suggested guiding law (6.54), as shown in Figure 6.5(a). The missile acceleration ( $\text{m/s}^2$ ) trajectory convergence is shown in Figure 6.5(b). Figure 6.5(c) shows the sliding surface trajectory convergence.

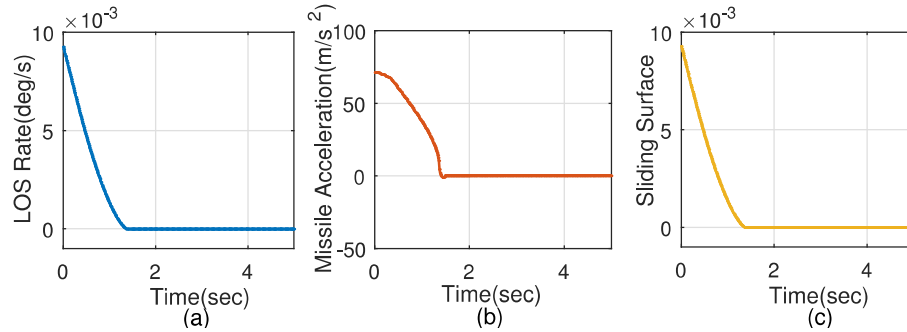


Figure 6.5: Trajectories in the case of non-maneuvering target (a) LOS rate; (b) Control input; (c) Sliding Surface.

**Case 2: Maneuvering target**

In this case, acceleration is considered as  $a_{T\theta} = 30 \text{ m/s}^2$ . Using the suggested guiding law (6.54), the LOS rate converges to zero or a small neighborhood of zero in finite-time, as illustrated in Figure 6.6(a). Figure 6.6(b) depicts the convergence of missile acceleration ( $\text{m/s}^2$ ) trajectory. The convergence of sliding surface trajectory is depicted in Figure 6.6(c).

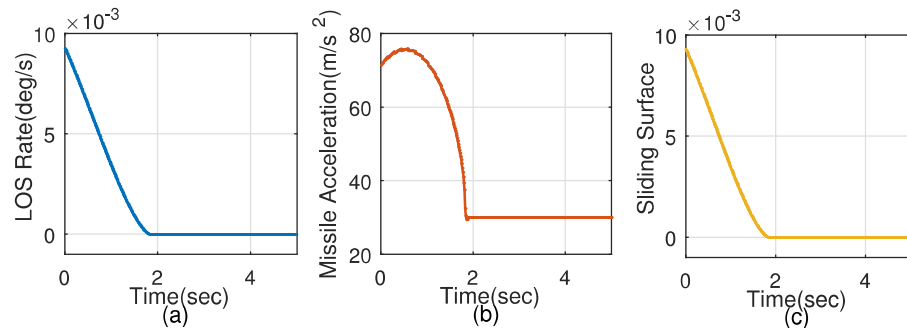


Figure 6.6: Trajectories in the case of maneuvering target (a) LOS rate; (b) Control input; (c) Sliding Surface.

Figure 6.7(a) and (b) shows the inter-event time  $T_k$  for non-maneuvering target and maneuvering target, respectively. From the Figures 6.7(a) and (b), one can see that the control utilization is minimum and it is updated whenever event occurs. Based on the above discussion and simulation results, we can conclude that the LOS rate is converging as intended.

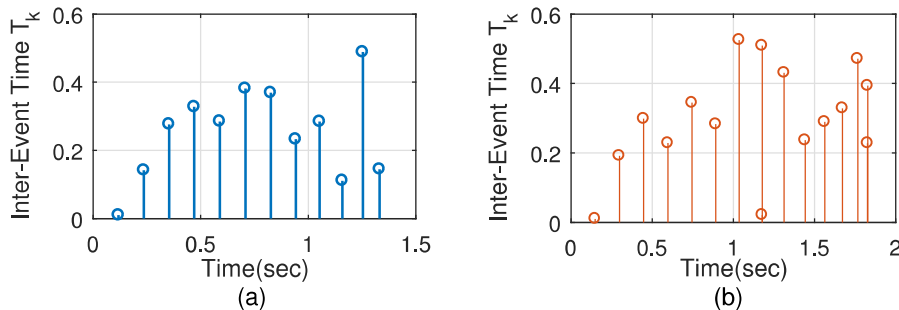


Figure 6.7: Inter-event Time  $T_k$  (a) Non-maneuvering target; (b) Maneuvering target.

## 6.5 Guidance Law with Predefined-Time Convergence

Consider a general class of nonlinear system represented as

$$\dot{x} = g(x, t), \quad x(0) = x_0, \quad x \in \mathbb{R}^n \quad (6.95)$$

where  $g : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$  is a continuous function on  $D \subseteq \mathbb{R}^n$  and  $x$  is the state of the system.

Predefined-time convergence property holds for the system (6.95) if it is fixed-time stable and any solution  $x(t, t_0, x_0)$  of the system (6.95) converges to zero in a predefined amount of time. Further, it is important to mention that the convergence time of the system can be chosen as per will of the designer.

Now, a lemma is introduced hereafter, which will be used further in stability analysis of the system with the guidance law to be designed.

**Lemma 10** [37]: *For the nonlinear system (6.95), if there exists a positive-definite, smooth, radially unbounded function  $V(x)$  such that*

$$\frac{\partial V}{\partial x} g(x, t) \leq -k \frac{(1 - e^{-V})}{(t_s - t)}, \quad x \neq 0, \quad t_0 \leq t < t_s \quad (6.96)$$

where  $k \in \mathbb{R}_{>1}$ , then, the predefined-time convergence property holds for the system (6.95), with  $t_s$  being the desired settling/convergence time.

### 6.5.1 Design of Guidance Law

In this part, the design of guidance law with predefined-time convergence is introduced for the planar relative motion. Then, the Lyapunov stability analysis is done for the resulting system which proves the predefined-time convergence of the LOS rate.

Recall the previous equation which has been given in (6.4) and can be represented as

$$\ddot{\theta} = -\frac{2\dot{r}}{r}\dot{\theta} + \frac{a_{T\theta}}{r} - \frac{a_{M\theta}}{r}. \quad (6.97)$$

Defining a new variable as  $x := \dot{\theta}$ , Eq. (6.97) can be represented as

$$\dot{x} = -\frac{2\dot{r}}{r}x - \frac{u}{r} + \frac{a_{T\theta}}{r}, \quad x \in \mathbb{R}. \quad (6.98)$$

Assume that the target acceleration  $a_{T\theta}$  satisfies the inequality

$$|a_{T\theta}| \leq d, \quad d > 0 \quad (6.99)$$

At first, the case is considered where the target acceleration is precisely known.

**Theorem 10** *For the system (6.98), if there exists  $k \in \mathbb{R}_{>1}$  such that the guidance law designed as*

$$u = \begin{cases} -\mathcal{N}\dot{r}x + a_{T\theta} + k\frac{(1-e^{-x})}{(t_s-t)}, & t_0 \leq t < t_s \\ -\mathcal{N}\dot{r}x + a_{T\theta}, & t \geq t_s \end{cases} \quad (6.100)$$

where  $\mathcal{N}$  represents the navigation ratio and its value is greater than two, then, the LOS rate converges to zero or remains within a small neighbourhood of zero within the predefined time  $t_s$ .

*Proof:* The proof of *Theorem 10* is discussed in two parts. The first part is considered for time  $t_0 \leq t < t_s$ . The second part is considered for time  $t \geq t_s$ .

**Part A:** For  $t_0 \leq t < t_s$ :

Suppose that the Lyapunov function is considered as

$$V_1 = \frac{1}{2}x^2 \implies |x| = \sqrt{2V_1}. \quad (6.101)$$

Taking the time derivative of (6.101), one obtains

$$\dot{V}_1 = x\dot{x} = -\frac{2\dot{r}}{r}x^2 - \frac{u}{r}x + \frac{a_{T\theta}}{r}x. \quad (6.102)$$

In view of (6.100), one can represent (6.102) as

$$\dot{V}_1 = \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2 - \frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)}. \quad (6.103)$$

From (6.101), one can represent (6.103) as

$$\dot{V}_1 = \frac{2(\mathcal{N} - 2)\dot{r}}{r}V_1 - \frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)}. \quad (6.104)$$

During the time span of the engagement process, the range satisfies the condition  $0 < r(t) < r(0)$  and its corresponding rate, i.e.,  $\dot{r}(t)$  remains less than zero for all  $t \geq 0$ .

Further, (6.104) becomes

$$\dot{V}_1 = -\frac{2(\mathcal{N} - 2)|\dot{r}|}{r}V_1 - \frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)} \leq -\frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)}. \quad (6.105)$$

The Eq. (6.105) can be modified in the form of the following inequality given by

$$\dot{V}_1 \leq -\frac{k}{r}|x| \frac{(1 - e^{-|x|})}{(t_s - t)} = \frac{k}{r}\sqrt{2V_1} \frac{(1 - e^{-\sqrt{2V_1}})}{(t_s - t)}.$$

Let us define another function as

$$V_2 = \sqrt{2V_1}.$$

Its time derivative becomes of the form

$$\dot{V}_2 = \frac{\sqrt{2}\dot{V}_1}{2\sqrt{V_1}} \leq -\frac{k}{r} \frac{(1 - e^{-V_2})}{(t_s - t)}. \quad (6.106)$$

Then, (6.106) becomes

$$\dot{V}_2 \leq -\frac{k}{r(0)} \frac{(1 - e^{-V_2})}{(t_s - t)} = -\eta \frac{(1 - e^{-V_2})}{(t_s - t)} \quad (6.107)$$

where  $\eta = \frac{k}{r(0)}$ .

In view of *Lemma 10*, (6.107) shows that the system exhibits the property of predefined-time stability as discussed in the previous section. Thus, the convergence of the LOS rate of the system (6.98) is obtained within the predefined time  $t_s$ .

**Part B:** For  $t \geq t_s$ :

In this region the predefined-time convergent term, i.e.,  $k \frac{(1 - e^{-x})}{(t_s - t)}$  is not present.

Now, (6.103) can be represented as

$$\dot{V}_1 = \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2. \quad (6.108)$$

In accordance with (6.101), one can write (6.108) as

$$\dot{V}_1 = \frac{2(\mathcal{N} - 2)\dot{r}}{r}V_1 = -\frac{2(\mathcal{N} - 2)|\dot{r}|}{r}V_1 \leq -\frac{2(\mathcal{N} - 2)|\dot{r}|}{r(0)}V_1 \leq 0. \quad (6.109)$$

Therefore, using (6.107), one obtains  $V_2 = \sqrt{2V_1} = 0$  which directly gives  $x = 0$  for all  $t \geq t_s$ . Since  $V_1$  is equal to  $x^2/2$ . This means that the LOS rate becomes zero for all  $t \geq t_s$ . Hence, the system (6.98) with the designed guidance law exhibits predefined-time stability for  $\eta = \frac{k}{r(0)} > 1$  which further implies  $k > r(0)$ . Hence, the proof of the *Theorem 10* is completed.  $\blacksquare$

Now, the case is considered where the target acceleration is not known but the information of its maximum upper bound is available.

**Theorem 11** *Suppose that the condition given by equation (6.99) is satisfied by the system (6.98). If there exists  $k \in \mathbb{R}_{>1}$  such that the guidance law designed as*

$$u = \begin{cases} -\mathcal{N}\dot{r}x + d\text{sign}(x) + k\frac{(1-e^{-x})}{(t_s-t)}, & t_0 \leq t < t_s \\ -\mathcal{N}\dot{r}x + d\text{sign}(x), & t \geq t_s \end{cases} \quad (6.110)$$

where  $\mathcal{N} > 2$  is the navigation ratio,  $d \geq |a_{T\theta}|$  is the gain,  $\text{sign}(\cdot)$  is signum function, then, the LOS rate becomes zero or remains within a small neighbourhood of zero within the predefined time  $t_s$ .

*Proof:* The proof of the *Theorem 11* is divided into two parts. The first part is given when time  $t_0 \leq t < t_s$ . The second part is given when time  $t \geq t_s$ .

**Part A:** For  $t_0 \leq t < t_s$ :

From the previous relation, the Lyapunov function is assumed as

$$V_3 = \frac{1}{2}x^2 \implies |x| = \sqrt{2V_3}. \quad (6.111)$$

Differentiating the above equation with respect to time, one gets

$$\dot{V}_3 = x\dot{x} = -\frac{2\dot{r}}{r}x^2 - \frac{u}{r}x + \frac{a_{T\theta}}{r}x. \quad (6.112)$$

In view of (6.110), The Eq. (6.112) can be represented as

$$\dot{V}_3 = \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2 - \frac{(d\text{sign}(x) - a_{T\theta})}{r}x - \frac{k}{r}x\frac{(1 - e^{-x})}{(t_s - t)}. \quad (6.113)$$

From the relation  $|a_{T\theta}| \leq d$ , (6.113) can be expressed in the form

$$\dot{V}_3 \leq \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2 - \frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)}. \quad (6.114)$$

Further, (6.114) can be expressed as

$$\dot{V}_3 \leq -\frac{(\mathcal{N} - 2)|\dot{r}|}{r}x^2 - \frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)} \leq -\frac{k}{r}x \frac{(1 - e^{-x})}{(t_s - t)} \leq -\frac{k}{r}|x| \frac{(1 - e^{-|x|})}{(t_s - t)}. \quad (6.115)$$

Using (6.111), i.e.,  $|x| = \sqrt{2V_3}$ , (6.115) can be represented in the form

$$\dot{V}_3 \leq -\frac{k}{r}\sqrt{2V_3} \frac{(1 - e^{-\sqrt{2V_3}})}{(t_s - t)}. \quad (6.116)$$

Let us consider a new function

$$V_4 = \sqrt{2V_3}.$$

Its time derivative becomes

$$\dot{V}_4 = \frac{\sqrt{2}\dot{V}_3}{2\sqrt{V_3}} \leq -\frac{k}{r} \frac{(1 - e^{-V_4})}{(t_s - t)} \leq -\frac{k}{r(0)} \frac{(1 - e^{-V_4})}{(t_s - t)} = -\eta \frac{(1 - e^{-V_4})}{(t_s - t)} \quad (6.117)$$

where  $\eta = \frac{k}{r(0)}$ .

In view of *Lemma 10*, The Eq. (6.117) shows that the system exhibits the property of predefined-time stability as discussed in the previous section. Therefore, the convergence of the LOS rate of the system (6.98) is obtained within the predefined time  $t_s$ .

**Part B:** For  $t \geq t_s$ :

In this region the predefined-time convergent term, i.e.,  $k \frac{(1 - e^{-x})}{(t_s - t)}$  is not present.

Now, (6.113) becomes

$$\begin{aligned} \dot{V}_3 &= \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2 - \frac{(d\text{sign}(x) - a_{T\theta})}{r}x \leq \frac{(\mathcal{N} - 2)\dot{r}}{r}x^2 = -\frac{(N - 2)|\dot{r}|}{r}x^2 \\ \dot{V}_3 &\leq -\frac{2(\mathcal{N} - 2)|\dot{r}|}{r}V_3. \end{aligned} \quad (6.118)$$

In view of (6.118), it is clear that the remaining part of the proof can be completed in exactly the same way as in the proof of *Theorem 10*. Hence the proof for  $t \geq t_s$  is omitted here.

From the aforementioned discussion, using (6.117), one gets  $V_4 = \sqrt{2V_3} = 0$  which implies  $x = 0 \forall t \geq t_s$  since  $V_3 = \frac{x^2}{2}$ . Thus, the LOS rate converges to zero within the predefined time  $t_s$ . Therefore, the predefined-time convergence of the LOS rate is achieved with the designed guidance law. The proof is completed. ■

**Remark 14** *The constant  $k$  in equation 6.110 can be interpreted as the control gain. It is important to note that  $k$  should be greater than 1 to achieve predefined-time convergence. In fact, opting for a higher value of  $k$  results in a stronger control, leading to faster convergence. Consequently, by selecting a higher value of  $k$ , convergence could be achieved well before  $t_s$ . A prudent approach would involve balancing this trade-off between faster convergence and increased control effort, and then accordingly selecting the gain  $k$ .*

One can observe that the guidance law (6.110) contains signum function which results in discontinuity in the guidance system, leading to the undesired chattering effect. For minimizing the chattering effect, various techniques have been used in the literature. One of the various techniques is the approximation of the discontinuous signum function by the class of smooth functions. Here, the function  $sign(\cdot)$  is approximated by the smooth saturation function  $sat_\omega(\cdot)$  as follows [125]:

$$sat_\omega(\xi) = \begin{cases} \frac{\xi}{\omega}, & |\xi| \leq \omega \\ 1, & \xi > \omega \\ -1, & \xi < -\omega \end{cases} \quad (6.119)$$

where  $\omega$  is a small positive constant.

## 6.5.2 Simulation results

This section demonstrates the performance of the designed guidance law through simulation by considering a practical example, as noted in the section 6.3.3. The missile's initial position is selected as  $x_{M0} = 0$  m and  $y_{M0} = 0$  m.  $V_M = 300$  m/sec. and  $\phi_M = 60^\circ$  are the initial velocity of the missile and the flight-path angle, respectively. The target's initial locations are chosen as  $x_{T0} = 5000$  m and  $y_{T0} = 5000$  m. The starting flight-path angle and velocity of the target are  $\phi_T = 120^\circ$  and  $V_T = 200$  m/sec., respectively. In simulations, the proposed approach is considered for the two cases as mentioned in the previous discussion. In the first case, the target acceleration is known and the second case considers that the target's acceleration is not available but its upper bound is given.

### ***Case 1: The target acceleration is exactly known***

The acceleration of the target is taken as  $a_{T\theta} = 20$  m/sec.<sup>2</sup>. For the proposed guidance

law (6.100), the design parameters are selected as  $\mathcal{N} = 3$  and  $k = 11000$  for which the results are depicted in Figures. 6.8 and 6.9. For  $t_s = 3$  sec., the LOS rate converges to zero or remain within a closed boundary of zero within the predefined time  $t_s$  (settling time), as shown in Figure 6.8(a). The missile acceleration ( $\text{m/sec.}^2$ ) (guidance law) is shown in Figure 6.9(a). Similarly, for  $t_s = 4$  sec., the LOS rate converges to zero within the predefined time  $t_s$ , as depicted in Figure 6.8(b). The missile acceleration ( $\text{m/sec.}^2$ ) (guidance law) is illustrated in Figure 6.9(b).

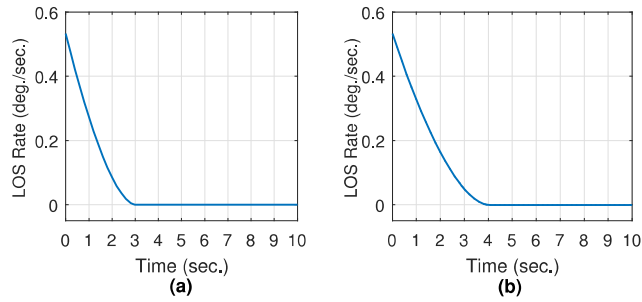


Figure 6.8: Convergence of the LOS rate in the chosen predefined time  $t_s = 3$  sec. and 4 sec., respectively.

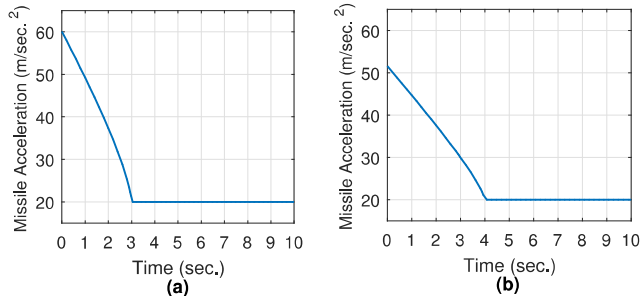


Figure 6.9: Missile acceleration tracking the target acceleration in the chosen predefined time  $t_s = 3$  sec. and 4 sec., respectively.

***Case 2: The target acceleration is not known***

Consider the case where only the maximum upper bound of the target acceleration is available. Suppose that it takes the value  $a_{T\theta} = 10 \text{ m/sec.}^2$ . For the guidance law (6.110), the design parameters are selected as  $N = 3$ ,  $d = 11$ ,  $\omega = 0.001$  and  $k = 25000$  for which the results are shown in Figures. 6.10 and 6.11. For  $t_s = 3$  sec., the LOS rate converges to zero or remains within a closed boundary of zero within the predefined time  $t_s$  (settling time), as depicted in Figure 6.10(a). The missile acceleration ( $\text{m/sec.}^2$ ) (guidance law) is

shown in Figure 6.11(a). Similarly, for  $t_s = 4$  sec., the LOS rate converges to zero within the predefined time  $t_s$ , as illustrated in Figure 6.10(b). The missile acceleration ( $\text{m}/\text{sec.}^2$ ) (guidance law) is illustrated in Figure 6.11(b).

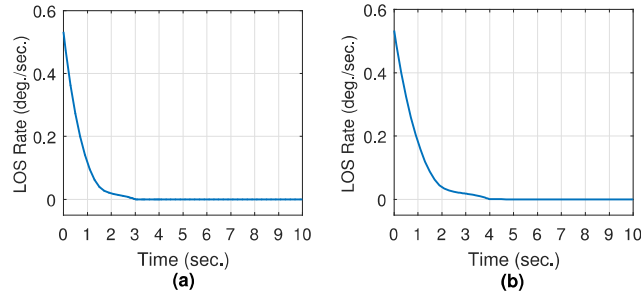


Figure 6.10: Convergence of the LOS rate in the chosen predefined time  $t_s = 3$  sec. and 4 sec., respectively.

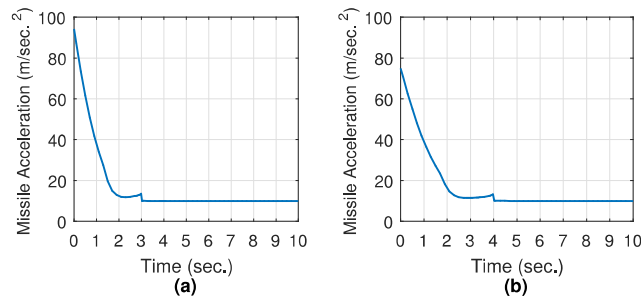


Figure 6.11: Missile acceleration tracking the target Acceleration in the chosen predefined time  $t_s = 3$  sec. and 4 sec., respectively.

From the discussions mentioned above in Figures. 6.8(a) and 6.8(b), it is observed that the LOS rate converges to zero or remain within a closed boundary of zero within the predefined time  $t_s = 3$  sec. and 4 sec. In Figures. 6.9(a) and 6.9(b), one can observe that the missile acceleration tracks the target acceleration within the chosen time  $t_s = 3$  sec. and 4 sec. for the case when target acceleration is precisely known. Similarly, in Figures. 6.10(a) and 6.10(b), it is noticed that the LOS rate converges to zero within the predefined time  $t_s = 3$  sec. and 4 sec. In Figures. 6.11(a) and 6.11(b), one can show that the missile acceleration tracks the target acceleration within the chosen time  $t_s = 3$  sec. and 4 sec. for the case when the target acceleration is not known but the upper bound of the target acceleration is available. In both cases, for the same initial condition, it can be seen that the LOS rate converges to zero or a small neighborhood of zero, and the missile

acceleration tracks the target acceleration within the chosen time  $t_s = 3$  sec. and 4 sec. Therefore, in the both cases, one can observe that the convergence time of the LOS rate in the guidance system can be chosen as per will of the designer. In Figures 6.11(a) and 6.11(b), the missile acceleration is continuous with respect to the state (LOS rate) but not with respect to time.

## 6.6 Summary

The guidance law is developed using various approaches. Firstly, the adaptive super-twisting algorithm is used to design guidance laws for the missile-target motion in a plane. Additionally, estimated information about the target acceleration obtained by an extended state observer is utilized. With this approach, the guidance law achieves LOS convergence in finite time. Secondly, the ETASTA-based guidance law is developed, where the LOS rate converges to zero within a finite time. A triggering condition is offered that uses the minimum amount of control and meets the stability requirements. Furthermore, a Zeno-free study is presented by providing a positive lower bound on the inter-event time. Lastly, the design of guidance law with predefined-time convergence is also proposed. With the proposed guidance law, the LOS rate converges to zero within a predefined time. Using this technique, the settling time of the LOS rate of the missile guidance system can be chosen at the discretion of the designer. The efficacy of the proposed techniques is illustrated via a practical example.

The upcoming chapter will provide a conclusion to the work presented in this thesis and offer a forecast of potential future endeavors.