Chapter 3

Inductive algebras for compact groups

In this chapter we show that *inductive algebras for a compact group are self-adjoint.* This is significant because, in general, the classification of self-adjoint inductive algebras is easier than the classification of all inductive algebras. This is because the methods of spectral theory are available only in the former case.

In Section 3.1 we prove some results about subalgebras of $L^\infty(X,\mu)$, which will be used in the proof of our main theorem, but which are also of independent interest.

3.1 Subalgebras of L^∞

Theorem 3.1.1. *Let (X,μ) be a measure space. The algebra $L^\infty(X,\mu)$ is finite dimensional if and only if all of its subalgebras are self-adjoint.*

Proof. Assume first that $L^\infty(X,\mu)$ is finite dimensional. Observe that under this hypothesis, if $f \in L^\infty(X,\mu)$, then there exists a simple function s such that $f = s$ almost everywhere (see [14, Prop 3.4.2] and [15, §13.3, Cor. 6]).

Let $\mathcal{A} \subseteq L^\infty(X,\mu)$ be a subalgebra. Let $\{f_1, f_2, \ldots, f_n\}$ be a basis for $\mathcal{A}$, and choose simple functions $s_1, s_2, \ldots, s_n$ such that $f_j = s_j$ (a.e.), $j = 1, 2, \ldots, n$. Define a map $\mathbf{s} : X \to \mathbb{C}^n$ by

$$\mathbf{s}(x) = (s_1(x), s_2(x), \ldots, s_n(x)) .$$

Since simple functions attain only finitely many values, $\mathbf{s}(X)$ is finite, and we may write

$$\mathbf{s}(X) \setminus \{0\} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\},$$

for some $m \in \mathbb{N}$.

Put $A_0 = \mathbf{s}^{-1}(0)$ and

$$A_k = \mathbf{s}^{-1}(\mathbf{v}_k), \qquad k = 1, 2, \ldots, m.$$

Then $\{A_k\}_{k=0}^m$ are disjoint, and

$$X = \bigcup_{k=0}^m A_k.$$

Let v_{kj} denote the j-th component of the vector $\mathbf{v}_k$. Observe that if $x \in A_k$ then $s_j(x) = v_{kj}$, $j = 1, \ldots, n$, $k = 1, \ldots, m$, i.e., each s_j is constant on each A_k. Therefore

$$s_j \in \operatorname{span}\{\chi_{A_k}\}_{k=1}^m, \quad j = 1, \ldots, n.$$

Therefore $\mathcal{A} \subseteq \operatorname{span}\{\chi_{A_k}\}_{k=1}^m$.

Fix distinct $h, k \in \{1, \ldots, m\}$. Since $\mathbf{v}_h \neq \mathbf{v}_k$, there exists $j = j(h,k)$ such that $v_{hj} \neq v_{kj}$. Since $\mathbf{v}_h \neq 0$, there exists $l = l(h)$ such that $v_{hl} \neq 0$. Observe that

$$\varphi_{hk} = (s_j - v_{kj}) s_l \in \mathcal{A}.$$

If $x \in A_k$, then

$$\begin{aligned}\varphi_{hk}(x) &= (s_j(x) - v_{kj})s_l(x) \\ &= (v_{kj} - v_{kj})v_{kl} \\ &= 0,\end{aligned}$$

and if $x \in A_h$, then

$$\begin{aligned}\varphi_{hk}(x) &= (s_j(x) - v_{kj})s_l(x) \\ &= (v_{hj} - v_{kj})s_l(x) \\ &= (v_{hj} - v_{kj})v_{hl} \\ &\neq 0.\end{aligned}$$

Put

$$\psi_{hk} = \frac{\varphi_{hk}}{(v_{hj} - v_{kj})v_{hl}}.$$

Then $\psi_{hk} \in \mathcal{A}$, $\psi(x) = 1$ if $x \in A_h$ and $\psi(x) = 0$ if $x \in A_k$.

Since $\chi_{A_h} = \prod_{k\neq h} \psi_{hk}$, it follows that $\chi_{A_h} \in \mathcal{A}$, $h = 1, \dots, m$. Therefore $\mathcal{A} = \operatorname{span}\{\chi_{A_k}\}_{k=1}^m$. Therefore $\mathcal{A}$ is self-adjoint.

Assume now that $L^\infty(X, \mu)$ is infinite dimensional. We claim first that X has a sequence $\{E_n\}_{n=1}^\infty$ of disjoint measurable subsets of positive measure. Indeed, there exists a real valued function $f \in L^\infty(X, \mu)$ such that

$$-\infty < \operatorname{ess\,inf} f < \operatorname{ess\,sup} f < \infty.$$

Put

$$c = \frac{\operatorname{ess\,inf} f + \operatorname{ess\,sup} f}{2}.$$

Let

$$Y = \{x \in X | f(x) > c\}, \quad \text{and} \quad Z = \{x \in X | f(x) \leq c\}.$$

Then Y and Z are disjoint measurable sets, and by the definitions of essential supremum and essential infimum $\mu(Y) > 0$ and $\mu(Z) > 0$. Since

$$L^\infty(X,\mu) \cong L^\infty(Y,\mu) \oplus L^\infty(Z,\mu),$$

either $L^\infty(Y,\mu)$ or $L^\infty(Z,\mu)$ must be infinite dimensional, say $\dim L^\infty(Z,\mu) = \infty$. Let $E_1 = Y$. Since $\dim L^\infty(Z,\mu) = \infty$, there exists a real-valued function $f_1 \in L^\infty(Z,\mu)$ such that

$$-\infty < \operatorname{ess\,inf} f_1 < \operatorname{ess\,sup} f_1 < \infty.$$

Let

$$Y_1 = \{x \in X | f_1(x) > c\}, \quad \text{and} \quad Z_1 = \{x \in X | f_1(x) \leq c\}.$$

Then Y_1 and Z_1 are disjoint measurable sets, and by the definitions of essential supremum and essential infimum $\mu(Y_1,\mu) > 0$ and $\mu(Z_1,\mu) > 0$. Since

$$L^\infty(Z,\mu) \cong L^\infty(Y_1,\mu) \oplus L^\infty(Z_1,\mu),$$

either $L^\infty(Y_1,\mu)$ or $L^\infty(Z_1,\mu)$ must be infinite dimensional, say $\dim L^\infty(Z_1,\mu) = \infty$. Let $E_2 = Y_1$. Because $E_2 \subseteq Z$, it follows that E_1 and E_2 are disjoint. In this way, we produce a sequence of disjoint measurable sets $E_1, E_2, \ldots$, each of positive measure.

Let

$$\tilde{\mathcal{A}} = \{f \in L^\infty(X,\mu) \mid f \text{ is a constant on } E_n,\ n = 1, 2, \ldots\}.$$

If $f \in \tilde{A}$, let $c_n(f)$ be the value of f on E_n.

Let

$$\mathcal{A} = \left\{ f \in \tilde{\mathcal{A}} \;\middle|\; \lim_{m\to\infty} \frac{c_{2m+1}(f) - c_1(f)}{(1/m)} = i \lim_{m\to\infty} \frac{c_{2m}(f) - c_1(f)}{(1/m)} \right\}.$$

It is easy to check that $\mathcal{A}$ is a subalgebra of $L^\infty(X,\mu)$. Now, define $f : X \to \mathbb{C}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in E_1, \\ \frac{1}{n} & \text{if } x \in E_n,\ n \text{ even}, \\ \frac{i}{n} & \text{if } x \in E_n,\ n > 1 \text{ odd}. \end{cases}$$

Then $f \in \mathcal{A}$ but $\overline{f} \notin \mathcal{A}$. Therefore, $\mathcal{A}$ is not self-adjoint. □

3.2 Compact groups

Theorem 3.2.1. *Let G be a compact group and π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a π-inductive algebra, then $\mathcal{A}$ is self-adjoint.*

Proof. By the Peter-Weyl theorem, $\mathcal{H}$ is finite dimensional.

Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathcal{A}$ (the nilradical of $\mathcal{A}$). Let

$$\mathcal{K} = \{x \in \mathcal{H} \mid Tx = 0, \quad \forall T \in \mathcal{N}\}.$$

By Engel's theorem [2, A.16], $\mathcal{K} \neq 0$. Observe that $\mathcal{N}$ is normalized by $\pi(G)$, so $\mathcal{K}$ is $\pi(G)$-invariant. However, since π is irreducible, it follows that $\mathcal{K} = \mathcal{H}$, whence $\mathcal{N} = 0$.

Let $\mathcal{A}^*$ denote the space of linear functionals on $\mathcal{A}$. For each $\lambda \in \mathcal{A}^*$, let

$$\mathcal{H}_\lambda = \{v \in \mathcal{H} \mid Tv = \lambda(T)v \quad \text{for all } T \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is abelian, and $\mathcal{N} = 0$, the Jordan-Chevalley decomposition [3, §4.2] implies that

$$\mathcal{H} = \bigoplus_{\lambda \in \mathcal{A}^*} \mathcal{H}_\lambda. \tag{3.1}$$

Since $\mathcal{H}$ is finite dimensional, all except a finite number of $\mathcal{H}_\lambda$ are zero. So

$$\Lambda = \{\lambda \in \mathcal{A}^* \mid \mathcal{H}_\lambda \neq 0\}$$

is a finite set.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of $\mathcal{H}$. There exists an inner product $\langle \cdot, \cdot \rangle_1$ on $\mathcal{H}$ such that $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_1$ if $\lambda \neq \mu$. Let σ denote the Haar probability measure on the compact group G. By Schur's lemma (see [18]), there exists a constant c such that

$$\langle v, w \rangle = c \int_G \langle \pi(g)v, \pi(g)w \rangle_1 \, d\sigma.$$

If $g \in G$ and $\lambda \in \mathcal{A}^*$, define $g\lambda : \mathcal{A} \to \mathbb{C}$ by

$$g\lambda(T) = \lambda(\pi(g)^{-1} T \pi(g)), \quad T \in \mathcal{A}.$$

This defines an action of G on $\mathcal{A}^*$, which preserves Λ.

Note that for any $g \in G$, $\lambda \in \mathcal{A}^*$ and $v \in \mathcal{H}_\lambda$, $\pi(g)v \in \mathcal{H}_{g\lambda}$. Also $\lambda \neq \mu$ implies $g\lambda \neq g\mu$. Therefore, if $\lambda \neq \mu$, $v \in \mathcal{H}_\lambda$ and $w \in \mathcal{H}_\mu$, then

$$\begin{aligned} \langle v, w \rangle &= c \int_G \langle \pi(g)v, \pi(g)w \rangle_1 \, d\mu \\ &= 0. \end{aligned}$$

Therefore $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$ if $\lambda \neq \mu$.

Observe that if $\lambda \in \Lambda$, then λ is multiplicative. Indeed, if $\lambda \in \Lambda$, then there exists $v \in \mathcal{H}_\lambda \setminus \{0\}$. Therefore, if $T_1, T_2 \in \mathcal{A}$, then

$$\lambda(T_1T_2)v = T_1T_2v = T_1(\lambda(T_2)v) = \lambda(T_2)T_1v = \lambda(T_2)\lambda(T_1)v.$$

Since $v \neq 0$, it follows that $\lambda(T_1T_2) = \lambda(T_1)\lambda(T_2)$.

It follows that the map $\mathcal{G} : \mathcal{A} \to L^\infty(\Lambda)$ (with respect to counting measure) defined by

$$\left[\mathcal{G}(T)\right](\lambda) = \lambda(T).$$

is an algebra homomorphism.

Since Λ is finite, $L^\infty(\Lambda)$ is finite dimensional, and so $\mathcal{G}(\mathcal{A})$ is self-adjoint by Theorem 3.1.1.

Let $T \in \mathcal{A}$. Then there exists $T_1 \in \mathcal{A}$ such that $\mathcal{G}(T_1) = \overline{\mathcal{G}(T)}$, i.e., $\lambda(T_1) = \overline{\lambda(T)}$ for all $\lambda \in \Lambda$. We claim that $T_1 = T^*$, i.e., that

$$\langle Tv, w\rangle = \langle v, T_1w\rangle \quad \text{for all } v, w \in \mathcal{H}.$$

By (3.1) it suffices to check this assuming that $v \in \mathcal{H}_\lambda$ and $w \in \mathcal{H}_\mu$ for $\lambda, \mu \in \Lambda$.

If $\lambda = \mu$, then

$$\begin{aligned}\langle Tv, w\rangle &= \langle \lambda(T)v, w\rangle \\ &= \langle v, \overline{\lambda(T)}w\rangle \\ &= \langle v, T_1w\rangle.\end{aligned}$$

If $\lambda \neq \mu$, then $\langle v, w \rangle = 0$, and so

$$\begin{aligned} \langle Tv, w \rangle &= \langle \lambda(T)v, w \rangle \\ &= 0, \quad \text{and} \\ \langle v, T_1 w \rangle &= \langle v, \mu(T_1)w \rangle \\ &= 0. \end{aligned}$$

□

Corollary 3.2.2. *Let G be a finite group and π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a π-inductive algebra, then $\mathcal{A}$ is self-adjoint.*

In view of Raghavan's theorem [8], it might appear that Corollary 3.2.2 may be used whenever Theorem 3.2.1 is applicable. However, that is not the case. Indeed, if $G = \mathrm{O}(2)$, the group of orthogonal 2×2 matrices, then G is compact and not abelian, but its group of components is abelian. If π is an irreducible representation of G of dimension greater than one, then Theorem 3.2.1 implies that all π-inductive algebras are self-adjoint, but Corollary 3.2.2 is not applicable.