Chapter 2

On the Existence of Simple Waves for Two-Dimensional Non-ideal Magnetohydrodynamics

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2.1 Introduction

Simple waves are incredibly versatile, contributing to various fields, including medical procedures such as lithotripsy, traffic flow analysis, and oceanographic studies. This demonstrates its broad utility for understanding and predicting wave behaviour in numerous scientific and engineering disciplines. In the realm of fluid dynamics, they are particularly useful in studying compressible flows, where there are significant changes in pressure (p) and density (ρ). Generally, a simple wave can be defined as a flow that depends solely on a single parameter. This single parameter is known as “phase". In order to get solutions to the flow problems, a simple wave plays an important role [2, 23–29]. Researchers often use simple wave analysis to study the influence of co-volume effects on compressible fluid dynamics and magnetic field evolution in magnetohydrodynamic (MHD) systems. This aids in developing and optimising MHD technologies, such as plasma confinement in fusion reactors or space plasma interactions [85–87].

To obtain solutions for a system of hyperbolic conservation laws, we know that the presence of Riemann invariants is very expressive [88]. Consider a 2×2 homogeneous first-order quasilinear evolution equation.

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{2.1.1}$$

where $\mathbf{U} = (U,\ V)$ and $\mathbf{F}(\mathbf{U}) = (f_1(\mathbf{U}),\ f_2(\mathbf{U}))$. Equation (2.1.1) can be expressed as $R_{1t} + \lambda_1 R_{1x} = 0, \quad R_{2t} + \lambda_2 R_{2x} = 0$, where $(R_1,\ R_2)$ is the Riemann invariant and the eigenvalues of the system are λ_1 and λ_2 [89, 90]. Invariance of $(R_1,\ R_2)$ leads to the conclusion that any flow of hyperbolic kind adjacent to a region of constant state is a simple wave [2]. However, in general theory, there are certain limitations to finding Riemann invariants. Moreover, the Riemann invariant need not exist for more than 2×2 systems [30]. To deal with simple waves arising from a system of conservation laws of hyperbolic nature, we mathematically use a powerful method known as characteristic decomposition. The characteristic decomposition method breaks down the solution into a combination of simple waves. This method is often used in computational fluid dynamics to solve fluid flow problems. This decomposition is used to prove the development of singularities [88] and to form the D’Alembert formula. In addition, this method leads to an analysis of the interaction of rarefaction waves governed by two-dimensional Euler equations [24, 91, 92]. In studying the interaction of simple waves, this method guides us to stay away from the mathematical challenges of the hodograph transformation interconnected with simple waves and their boundaries [24, 91, 93].

Dai and Zhang [31] were the first to introduce this method for the pressure gradient system. The study was extended for ideal gases by Li et al. [32], for non-ideal gases by Zafar and Sharma [33, 34], and for a class of pressure laws by Xiao and Li [35] in two-dimensional Euler equations for compressible flows. In view of this existing literature, our work offers insights into how the real gas influences the behaviour of the local structure of the self-similar solutions of Euler's equations in the presence of a magnetic field. It has wonderful applications to solve a GRP in compressible fluid flows [94, 95]. In fact, it is essential to construct patches of globally smooth solutions and to offer a route to obtain a priori estimates of solutions [96–101]. For a quasilinear strictly hyperbolic system, Hu and Sheng [27, 28] establish a nice sufficient condition to ensure characteristic decompositions. For a non-reducible system, C̀anic and Keyfitz [39] generalized the fundamental theorem [2]. Recently, using the sufficient condition provided by Hu and Sheng [27], Chen and Sheng [40], and Barthwal and Sekhar [41] extended the results in MHD systems. One of the main purposes of this chapter is to introduce the problem solved by Zafar and Sharma [33, 34] in MHD, which is a very complex medium for studying such wave phenomena. This work aims to generalize the well-known theorem of Courant and Friedrichs [2] for a reducible system, as well as a motivating study by Li et al. [32] for ideal gases, to the pseudo-steady irrotational MHD system for the non-ideal gas. The non-ideal gas effects presented here are characterized by the co-volume equation of state, which generalizes the ideal gas. It is also known as Noble-Abel gas. This equation of state may be seen as a perfect fluid polluted by dusty particles [34]. This investigation demonstrates the famous result of Courant and Friedrichs [2], which states that *the flow adjacent to a region of constant state is simple* in a 2D MHD system.

In an ideal magnetofluid, the electric conductivity of the fluid tends to infinity while the viscosity coefficients and heat conductivity are ignored. According to Cabannes [23], the Euler equations that can be used to govern an ideal unsteady compressible magnetohydrodynamic system are as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{U}) = 0,\\
&\frac{\partial}{\partial t}(\rho \mathbf{U}) + \nabla.(\rho \mathbf{U} \otimes \mathbf{U} + pI) - \mu(\nabla \times \mathbf{H}) \times \mathbf{H} = 0,\\
&\frac{\partial}{\partial t}(\rho E + \frac{1}{2}\mu|\mathbf{H}|^2) + \nabla.(\rho \mathbf{U} E + \mathbf{U} p) - \nabla.\big(\mu(\mathbf{U} \times \mathbf{H}) \times \mathbf{H}\big) = 0,\\
&\frac{\partial}{\partial t}\mathbf{H} + \nabla \times (\mathbf{H} \times \mathbf{U}) = 0,\\
&\nabla \cdot \mathbf{H} = 0,
\end{aligned} \tag{2.1.2}$$

where the density is denoted by ρ, $\mathbf{U} = (U,\ V,\ W)$ denotes fluid's velocity. $E = e + \frac{1}{2}||\mathbf{U}||^2$ and $e = e(S,\ \rho)$ are known as the total energy and internal energy, respectively. Here S denotes the specific entropy per unit mass, which is conserved along particle paths in the absence of dissipative effects. Magnetic permeability μ is constant, and $\mathbf{H} = (H_1,\ H_2,\ H_3)$ denotes the magnetic field vector, respectively. Let us suppose variables (U, V, ρ, p, H) are independent of the space variable z, which typically denotes the out-of-plane direction in three-dimensional flows. Here, p denotes the thermodynamic pressure given by the equation of state. Also, we consider $\mathbf{H} = (0,\ 0,\ H)$ and $\mathbf{U} = (U,\ V,\ 0)$, that is, the magnetofluid velocity and the magnetic field are orthogonal to each other. On the combination of the first and last equations of the above system of equations (2.1.2), we get

$$\frac{\partial}{\partial t}\left(\frac{H}{\rho}\right) + U\frac{\partial}{\partial x}\left(\frac{H}{\rho}\right) + V\frac{\partial}{\partial y}\left(\frac{H}{\rho}\right) = 0.$$

Moreover, it is obtained that $\frac{H}{\rho}$ is constant along each streamline. Let $\frac{H}{\rho}=\alpha_0$, be a positive constant. Under these considerations, system (2.1.2) has the following form

$$\begin{aligned}
&\rho_t+(\rho U)_x+(\rho V)_y=0,\\
&(\rho U)_t+\left(\rho U^2+p+\frac{\mu}{2}H^2\right)_x+(\rho UV)_y=0,\\
&(\rho U)_t+(\rho UV)_x+\left(\rho V^2+p+\frac{\mu}{2}H^2\right)_y=0,\\
&\left(\rho E+\frac{\mu}{2}H^2\right)_t+\left(\rho UE+Up+\mu UH^2\right)_x+\left(\rho VE+Vp+\mu VH^2\right)_y=0.
\end{aligned} \tag{2.1.3}$$

The structure of this chapter is organized as follows: Section 2.2 discusses the existence of simple waves in an isentropic and irrotational flow within a two-dimensional steady MHD system. In Section 2.3, the proof of simple wave existence for isentropic and irrotational flows in a two-dimensional pseudo-steady MHD system is presented. Additionally, Section 2.4 explores the application of this theory, extending these results to a comprehensive MHD system by assuming constant vorticity and entropy in the direction of pseudo-flow characteristics. The chapter concludes with Section 2.5, summarising the key findings.

2.2 Steady MHD 2D system

For compressible non-ideal gas, the system of 2-D Euler equations in steady case is given by

$$\rho\left(U_x+V_y\right)+U\rho_x+V\rho_y=0, \tag{2.2.1}$$

$$\rho\left(UU_x+VU_y\right)+\left(p_\rho+\mu HH_\rho\right)\rho_x=0, \tag{2.2.2}$$

$$\rho\left(UV_x+VV_y\right)+\left(p_\rho+\mu HH_\rho\right)\rho_y=0, \tag{2.2.3}$$

with the equation of state

$$p(\rho) = \mathcal{A}\frac{\rho^n}{(1-a\rho)^n}, \tag{2.2.4}$$

where $1 \leq n \leq 3$, $A > 0$ is a constant, and a is the van der Waals excluded volume, which is assumed to be a constant. From the combination of equations (2.2.2) and (2.2.3), we get

$$U^2U_x + V^2V_y + UV(U_y + V_x) + \left(\frac{p_\rho + \mu H H_\rho}{\rho}\right)(U\rho_x + V\rho_y) = 0. \tag{2.2.5}$$

We consider flow to be irrotational, i.e.,

$$U_y = V_x. \tag{2.2.6}$$

Using equation (2.2.1) and the irrotationality of flow in equation (2.2.5), we obtained

$$\left((p_\rho + \mu H H_\rho) - U^2\right)U_x - 2UVU_y + \left((p_\rho + \mu H H_\rho) - V^2\right)V_y = 0. \tag{2.2.7}$$

Equations (2.2.6) and (2.2.7) can be rewritten as

$$\begin{aligned} &\left(w^2 - U^2\right)U_x - 2UVU_y + \left(w^2 - V^2\right)V_y = 0,\\ &U_y - V_x = 0, \end{aligned} \tag{2.2.8}$$

where $w = (c^2 + b^2)^{\frac{1}{2}}$ is a characteristic speed that combines the sound speed (c) and Alfvén speed (b), and is called the magneto-acoustic speed. Here $c = \left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}$ and $b = (\mu\alpha_0^2\rho)^{1/2}$. The matrix form of system (2.2.8) can be presented as

$$\begin{bmatrix} U \\ V \end{bmatrix}_x + \begin{bmatrix} \frac{-2UV}{w^2-U^2} & \frac{w^2-V^2}{w^2-U^2} \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}_y = 0. \tag{2.2.9}$$

The determinantal equation of system (2.2.9) is

$$\lambda^2+\left(\frac{2UV}{w^2-U^2}\right)\lambda+\frac{w^2-V^2}{w^2-U^2}=0. \tag{2.2.10}$$

The eigenvalues and the corresponding eigenvectors of equation (2.2.10) are

$$\begin{aligned}\lambda_{\pm}&=\frac{UV\pm w\sqrt{U^2+V^2-w^2}}{U^2-w^2},\\ m_{\pm}&=\left[1,\lambda_{\mp}\right].\end{aligned} \tag{2.2.11}$$

The expressions for the eigenvalues $\lambda_{\pm}$ involve magneto-acoustic speed (w), sound speed (c) and Alfvén speed (b). These equations describe the behaviour of waves and disturbances in a magneto-acoustic system, considering the interaction of sound and magnetic waves. In addition, the hyperbolicity of the system is determined by the sign of $U^2+V^2-w^2$. If it is positive, the system is hyperbolic (supersonic) and the characteristic speeds are real. If it is zero, the system is parabolic (sonic). This means that the system (2.2.9) is of mixed type and changes its behaviour from hyperbolic to elliptic across the sonic boundary. Here, we consider only the hyperbolic case. The characteristic form of the system (2.2.9) is

$$m_{\pm}\begin{bmatrix}U_x\\ V_x\end{bmatrix}+\lambda_{\pm}m_{\pm}\begin{bmatrix}U_y\\ V_y\end{bmatrix}=0. \tag{2.2.12}$$

Equation (2.2.12) can be represented as

$$\partial_{\pm}U+\lambda_{\mp}\partial_{\pm}V=0, \tag{2.2.13}$$

where $\partial_{\pm} = \partial_x + \lambda_{\pm}\partial_y$. To prove the existence of a simple wave our aim is to compute

$$\begin{aligned}\partial_+\partial_- U &= \left(\partial_x + \lambda_+\partial_y\right)\left(\partial_x U + \lambda_-\partial_y U\right)\\ &= U_{xx} + \left(\lambda_+ + \lambda_-\right) U_{xy} + \lambda_+\lambda_- U_{yy} + \partial_+\lambda_- U_y.\end{aligned} \tag{2.2.14}$$

Using equation (2.2.13) we find expression of $\partial_+\lambda_-$ and it is given by

$$\begin{aligned}\partial_+\lambda_- &= \partial_U\lambda_-\partial_+ U + \partial_V\lambda_-\partial_+ V\\ &= \left(\partial_U\lambda_- - \frac{1}{\lambda_-}\partial_V\lambda_-\right)\partial_+ U.\end{aligned} \tag{2.2.15}$$

Note: In a similar fashion, we have the following expressions

$$\begin{aligned}\partial_-\lambda_{\pm} &= \left(\partial_U\lambda_{\pm} - \partial_V\lambda_{\pm}/\lambda_+\right)\partial_- U\\ \partial_+\lambda_{\pm} &= \left(\partial_U\lambda_{\pm} - \partial_V\lambda_{\pm}/\lambda_-\right)\partial_+ U.\end{aligned}$$

Also, from system (2.2.8) we can write

$$U_{xx} - \frac{2UV}{w^2 - V^2} U_{xy} + \frac{w^2 - U^2}{w^2 - V^2} U_{yy} = \frac{w^2 - V^2}{w^2 - U^2}\left[\left(\frac{2UV}{w^2 - V^2}\right)_x U_y - \left(\frac{w^2 - U^2}{w^2 - V^2}\right)_x U_x\right].$$

With the help of the characteristic equation (2.2.10), the above expression is reduced to the following form

$$U_{xx} + \left(\lambda_+ + \lambda_-\right) U_{xy} + \lambda_+\lambda_- U_{yy} = \frac{w^2 - V^2}{w^2 - U^2}\left[\left(\frac{2UV}{w^2 - V^2}\right)_x U_y - \left(\frac{w^2 - U^2}{w^2 - V^2}\right)_x U_x\right]. \tag{2.2.16}$$

Using expressions of $\partial_+\lambda_-$ and $U_{xx} + (\lambda_+ + \lambda_-)\,U_{xy} + \lambda_+\lambda_- U_{yy}$, from equations (2.2.15) and (2.2.16) respectively, equation (2.2.14) has taken the following form

$$\partial_+\partial_- U = \frac{w^2 - V^2}{w^2 - U^2}\left[\left(\frac{2UV}{w^2 - V^2}\right)_x U_y - \left(\frac{w^2 - U^2}{w^2 - V^2}\right)_x U_x\right] + \left(\partial_U \lambda_- - \frac{1}{\lambda_-}\partial_V \lambda_-\right)\partial_+ U. \tag{2.2.17}$$

To solve the RHS of equation (2.2.17), expressions for $(w^2)_x, (w^2)_U$ and $(w^2)_V$ are required, and can be given as

$$\begin{aligned}
\left(w^2\right)_x &= -\left[\frac{n - 1 + 2a\rho}{1 - a\rho} + \mu\alpha_0^2\frac{\rho}{w^2}\right]\left(UU_x + VV_y\right),\\
\left(w^2\right)_U &= -\left[\frac{n - 1 + 2a\rho}{1 - a\rho} + \mu\alpha_0^2\frac{\rho}{w^2}\right] U,\\
\left(w^2\right)_V &= -\left[\frac{n - 1 + 2a\rho}{1 - a\rho} + \mu\alpha_0^2\frac{\rho}{w^2}\right] V.
\end{aligned} \tag{2.2.18}$$

Now, using equation (2.2.18), we have

$$\begin{aligned}
\left(\frac{2UV}{w^2 - V^2}\right)_x = {} & \frac{2}{\left(w^2 - V^2\right)^2(1 - a\rho)}[VU_x\left(w^2 - V^2 + (n-1)\,U^2 + a\rho\left(2U^2 - w^2 + V^2\right)\right)\\
& + UU_y\left(w^2 + nV^2 + a\rho\left(V^2 - w^2\right)\right)] + \frac{\mu\alpha_0^2\rho}{w^2}\frac{2UV}{\left(w^2 - V^2\right)^2}\left(UU_x + VU_y\right),
\end{aligned} \tag{2.2.19}$$

$$\begin{aligned}
\left(\frac{w^2 - U^2}{w^2 - V^2}\right)_x = {} & \frac{2}{\left(w^2 - V^2\right)^2(1 - a\rho)}[-UU_x\left(2w^2 - (n+1)V^2 + (n-1)\,U^2 + 2a\rho\left(U^2 - w^2\right)\right)\\
& + VU_y\left(2w^2 - (n+1)V^2 + (n-1)U^2 + 2a\rho\left(V^2 - w^2\right)\right)]\\
& + \frac{\mu\alpha_0^2\rho}{w^2}\frac{\left(U^2 - V^2 + 1\right)}{\left(w^2 - V^2\right)^2}\left(UU_x + VU_y\right).
\end{aligned} \tag{2.2.20}$$

Differentiating equation (2.2.10) partially with respect to U and V in the direction of λ_-, we get

$$\begin{aligned}
\partial_U\lambda_- &= \frac{2U\lambda_-^2 - 2V\lambda_- + U\left(1+\lambda_-^2\right)\left(\frac{n-1+2a\rho}{1-a\rho} + \frac{\mu\alpha_0^2\rho}{w^2}\right)}{2\lambda_-\left(w^2-U^2\right)+2UV},\\
\partial_V\lambda_- &= \frac{2V - 2U\lambda_- + U\left(1+\lambda_-^2\right)\left(\frac{n-1+2a\rho}{1-a\rho} + \frac{\mu\alpha_0^2\rho}{w^2}\right)}{2\lambda_-\left(w^2-U^2\right)+2UV}.
\end{aligned} \tag{2.2.21}$$

On putting the values of $\partial_U\lambda_-$ and $\partial_V\lambda_-$ from the above equation (2.2.21) in equation (2.2.15), we have

$$\partial_+\lambda_- = \left[\left(\frac{n+1}{1-a\rho}\right) + \mu\alpha_0^2\frac{\rho}{w^2}\right]\frac{\left(U\lambda_- - V\right)^3}{w^2\lambda_-\left(2\lambda_-\left(w^2-U^2\right)+2UV\right)}\partial_+U. \tag{2.2.22}$$

Now, we are in a good position to calculate $\partial_+\partial_-U$. Therefore, with the help of equations (2.2.19), (2.2.20) and (2.2.22), equation (2.2.17) is reduced in the following form

$$\begin{aligned}
\partial_+\partial_-U = {} & \frac{1}{w^2(1-a\rho)(w^2-U^2)(w^2-V^2)}[U_x^2\{Uw^2(2w^2-(n+1)V^2+(n-1)U^2)\\
& - \mu\alpha_0^2(U^2-V^2+1)\} + U_xU_y\{2w^2V(w^2-V^2+(n-1)U^2) + 2\mu\alpha_0^2\rho U^2V\\
& - Yw^2(2w^2-(n+1)V^2+(n-1)U^2) - V\mu\alpha_0^2\rho(U^2-V^2+1) + Z\}\\
& + U_y^2\{2w^2U(w^2+nV^2) + 2\mu\alpha_0^2\rho UV^2 + \lambda_+Z\}\\
& + a\rho\{U_x^2\left[2Uw^2\left(U^2-w^2\right) + \mu\alpha_0^2\rho\left(U^2-V^2+1\right)\right]\\
& + U_y^2\left[2w^2U\left(V^2-w^2\right) - 2\mu\alpha_0^2\rho UV^2\right] + U_xU_y[2w^2V\left(2U^2-w^2+V^2\right)\\
& - 2\mu\alpha_0^2\rho VU^2 - 2Vw^2(V^2-w^2) + \mu\alpha_0^2\rho V(U^2-V^2+1)]\}],
\end{aligned} \tag{2.2.23}$$

where

$$Z = \frac{\left(w^2-U^2\right)\left(w^2-V^2\right)}{w^2\lambda_-}\frac{\left[(n+1)\,w^2+(1-a\rho)\,\alpha_0^2\right]\left(U\lambda_-V\right)^3}{\left[2\lambda_-\left(w^2-U^2\right)+2UV\right]}.$$

Equation (2.2.23) is in quadratic form. By factorising the equation (2.2.23) in the λ_- direction, we have

$$\partial_+\partial_-U = \frac{1}{w^2\left(1-a\rho\right)\left(w^2-U^2\right)\left(w^2-V^2\right)}[\{Uw^2\left(2w^2-\left(n+1\right)V^2+\left(n-1\right)U^2\right) \\ -\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\}\left(U_x+\alpha_1U_y\right) \\ +a\rho\{2Uw^2\left(U^2-w^2\right)+\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\left(U_x+\beta_1U_y\right)\}]\partial_-U,$$

where

$$\alpha_1 = \frac{2w^2U\left(w^2+nV^2\right)+2\mu\alpha_0^2\rho UV^2+\lambda_+Z}{\lambda_-\left[Uw^2\{2w^2-\left(n+1\right)V^2+\left(n-1\right)U^2\}-\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\right]},$$

and

$$\beta_1 = \frac{2w^2U\left(V^2-w^2\right)-\mu\alpha_0^2\rho UV^2}{\lambda_-\left[2w^2U\left(U^2-w^2\right)+\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\right]}.$$

Thus, we proved the following theorems.

Theorem 2.2.1. *For a non-ideal gas, the 2D isentropic irrotational steady compressible MHD system has the characteristic decomposition as follows:*

$$\partial_+\partial_-U = M\partial_-U,$$

where

$$M = \frac{1}{w^2\left(1-a\rho\right)\left(w^2-U^2\right)\left(w^2-V^2\right)}[\{Uw^2\left(2w^2-\left(n+1\right)V^2+\left(n-1\right)U^2\right) \\ -\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\}\left(U_x+\alpha_1U_y\right) \\ +a\rho\{2Uw^2\left(U^2-w^2\right)+\mu\alpha_0^2\rho\left(U^2-V^2+1\right)\left(U_x+\beta_1U_y\right)\}].$$

Similarly, we can show that

$$\partial_-\partial_+U = \overline{M}\partial_+U,$$

for some nice factor $\overline{M}$.

Therefore, by using the above theorem, we have the following result:

Theorem 2.2.2. *For an isentropic and irrotational flow governed by a 2D steady MHD system with a non-ideal gas, the region adjacent to the constant state will be a simple wave. In addition, the flow variables $(U,\ V,\ w)$ will be constant along the wave characteristics family, and that family has to be straight lines.*

2.3 Pseudo-steady MHD 2D system

Let us assume that the flow is smooth and isentropic and $u = U - \xi$ and $v = V - \eta$. The equation of the MHD system in the self-similar plane $(\xi, \eta) = \left(\frac{x}{t}, \frac{y}{t}\right)$ is given as

$$(\rho u)_\xi + (\rho v)_\eta + 2\rho = 0, \tag{2.3.1}$$

$$\frac{1}{\rho}\left(p_\rho + \mu H H_\rho\right)\rho_\xi + u u_\xi + v u_\eta + v = 0, \tag{2.3.2}$$

$$\frac{1}{\rho}\left(p_\rho + \mu H H_\rho\right)\rho_\eta + u v_\xi + v v_\eta + v = 0, \tag{2.3.3}$$

with the equation of state given in (2.2.4). By solving equations (2.3.2) and (2.3.3), we get

$$u^2 u_\xi + uv(u_\eta + v_\xi) + v^2 v_\eta + \Big(\frac{p_\rho + \mu H H_\rho}{\rho}\Big)(u\rho_\xi + v\rho_\eta) + (u^2 + v^2) = 0, \tag{2.3.4}$$

and the irrotationality condition in self-similar plane is given by

$$-u_\eta + v_\xi = 0. \tag{2.3.5}$$

Using $u = U - \xi$ and $v = V - \eta$ and with the help of equations (2.3.1), (2.3.5), equations (2.3.4) and (2.3.5) can be transformed into the following system.

$$\begin{aligned} &\left(w^2 - u^2\right) U_\xi - 2uvu_\eta + \left(w^2 - v^2\right) V_\eta = 0, \\ &- U_\eta + V_\xi = 0. \end{aligned} \tag{2.3.6}$$

The matrix form of the above system of equations (2.3.6) is

$$\begin{bmatrix} U \\ V \end{bmatrix}_\xi + \begin{bmatrix} \frac{-2uv}{w^2-u^2} & \frac{w^2-v^2}{w^2-u^2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}_\eta = 0. \tag{2.3.7}$$

The characteristic equation of the system of equations (2.3.7) is given as

$$\Lambda^2 + \left(\frac{2uv}{w^2 - u^2}\right) \Lambda + \frac{w^2 - v^2}{w^2 - u^2} = 0. \tag{2.3.8}$$

The characteristic roots and the corresponding left eigenvectors of the characteristic equation (2.3.8) are

$$\begin{aligned} &\frac{d\eta}{d\xi} = \Lambda_\pm = \frac{uv \pm w\sqrt{u^2 + v^2 - w^2}}{u^2 - w^2}, \\ &M_\pm = \left[1, \Lambda_\mp\right]. \end{aligned} \tag{2.3.9}$$

From the eigenvalues expression, it is clear that the system is supersonic for $q^2 > \omega^2$, subsonic for $q^2 < \omega^2$ and sonic for $q^2 = \omega^2$; where $q^2 = u^2 + v^2$. Likewise in steady case, we have

$$\partial_\pm U + \Lambda_\mp \partial_\pm V = 0, \tag{2.3.10}$$

where $\partial_\pm := \partial_\xi + \Lambda_\pm \partial_\eta$. Now, to compute $\partial_+\partial_- U$, we can write

$$\begin{aligned}\partial_+\partial_- U &= (\partial_\xi + \Lambda_+\partial_\eta)(\partial_\xi + \Lambda_-\partial_\eta)U \\ &= (\partial_\xi + \Lambda_+\partial_\eta)(\partial_\xi U + \Lambda_-\partial_\eta U) \\ &= U_{\xi\xi} + (\Lambda_+ + \Lambda_-)U_{\xi\eta} + \Lambda_+\Lambda_- U_{\eta\eta} + U_\eta \partial_+\Lambda_-. \qquad (2.3.11)\end{aligned}$$

Using the irrotationality condition (2.3.5) in characteristic equation (2.3.8), we get

$$U_{\xi\xi} + (\Lambda_+ + \Lambda_-)U_{\xi\eta} + (\Lambda_+\Lambda_-)U_{\eta\eta} = \frac{w^2 - v^2}{w^2 - u^2}\left[\left(\frac{2uv}{w^2 - v^2}\right)_\xi U_\eta - \left(\frac{w^2 - u^2}{w^2 - v^2}\right)_\xi U_\xi\right]. \qquad (2.3.12)$$

Our aim is to find the expression of $\partial_+\partial_- U$. For this we use equation (2.3.12) in equation (2.3.11),

$$\partial_+\partial_- U = \frac{w^2 - v^2}{w^2 - u^2}\left[\left(\frac{2uv}{w^2 - v^2}\right)_\xi U_\eta - \left(\frac{w^2 - u^2}{w^2 - v^2}\right)_\xi U_\xi\right] + U_\eta \partial_+\Lambda_-. \qquad (2.3.13)$$

Now, we calculate $\partial_+\Lambda_-$. Since $\Lambda_\pm = \Lambda_\pm(u, v, w^2)$ and $u = U - \xi$, $v = V - \eta$, thus we get

$$\partial_+\Lambda_- = \partial_u\Lambda_-\partial_+U + \partial_v\Lambda_-\partial_+V + \partial_{w^2}\Lambda_-\partial_+w^2 - (\partial_u\Lambda_+ + \Lambda_+\partial_v\Lambda_+). \qquad (2.3.14)$$

From the characteristic equation (2.3.8), we have

$$\partial_u\Lambda_\pm = \frac{\Lambda_\pm(u\Lambda_\pm - v)}{(w^2 - u^2)\Lambda_\pm + uv}. \qquad (2.3.15)$$

Following the similar calculation as in equation (2.3.15), we obtain

$$\partial_v\Lambda_\pm = -\frac{(u\Lambda_\pm - v)}{(w^2 - u^2)\Lambda_\pm + uv}. \qquad (2.3.16)$$

From equations (2.3.15) and (2.3.16), it can be shown that

$$\partial_u \Lambda_\pm + \Lambda_\pm \partial_v \Lambda_\pm = 0. \tag{2.3.17}$$

To find simple expressions of the RHS of equation (2.3.13), the expression of $(w^2)_\xi$ and $(w^2)_\eta$ is required and obtained as

$$(w^2)_\xi = -\Big[\frac{n-1+2a\rho}{1-a\rho} + \mu\alpha_0^2 \frac{\rho}{w^2}\Big](uU_\xi + vV_\xi),$$

and

$$(w^2)_\eta = -\Big[\frac{n-1+2a\rho}{1-a\rho} + \mu\alpha_0^2 \frac{\rho}{w^2}\Big](uU_\eta + vV_\eta).$$

Using above expressions of $(w^2)_\xi$ and $(w^2)_\eta$, we obtain

$$\begin{aligned}\partial_\pm w^2 &= (\partial_\xi + \Lambda_\pm \partial_\eta) w^2 \\ &= -\Big[\frac{n-1+2a\rho}{1-a\rho} + \mu\alpha_0^2 \frac{\rho}{w^2}\Big]\Big(u - \frac{v}{\Lambda_\mp}\Big)\partial_\pm U.\end{aligned} \tag{2.3.18}$$

Using equations (2.3.10) and (2.3.18) in equation (2.3.14), we get

$$\partial_+ \Lambda_- = \Big[\partial_u \Lambda_- - \frac{1}{\Lambda_+}\partial_u \Lambda_- - [\frac{n-1+2a\rho}{1-a\rho} + \mu\alpha_0^2 \frac{\rho}{w^2}](u - \frac{v}{\Lambda_+})\partial_{w^2}\Lambda_-\Big]\partial_+ U. \tag{2.3.19}$$

From characteristic equation (2.3.8), we can find

$$\partial_{w^2}\Lambda_- = -\frac{(u\Lambda_- - v)^2}{w^2[2\Lambda_-(w^2 - u^2) + 2uv]}. \tag{2.3.20}$$

Using equation (2.3.20) in equation (2.3.19), we get

$$\partial_+\Lambda_- = \frac{(u\Lambda_- - v)^3}{w^2\Lambda_-[2\Lambda_-(w^2-u^2)+2uv]}\left[2+\frac{n-1+2a\rho}{1-a\rho}+\mu\alpha_0^2\frac{\rho}{w^2}\right]\partial_+U - (\partial_u\Lambda_- + \Lambda_-\partial_v\Lambda_-). \tag{2.3.21}$$

By using the expressions of $(w^2)_\xi$, $(w^2)_\eta$ and $\partial_+\Lambda_-$, we obtain the following.

$$\begin{aligned}\left(\frac{2uv}{w^2-v^2}\right)_\xi = \frac{2}{(w^2-v^2)^2w^2(1-a\rho)}\Big[&vU_\xi\{w^2(w^2-v^2)(1-a\rho)+u^2\big[(n-1+2a\rho)w^2\\ &+\mu\alpha_0^2\rho(1-a\rho)\big]\}uU_\eta\{w^2(w^2-v^2)(1-a\rho)+v^2\big[(n-1+2a\rho)w^2\\ &+\mu\alpha_0^2\rho(1-a\rho)\big]+2v^2\}-v\{w^2(w^2-v^2)(1-a\rho)\}\Big],\end{aligned} \tag{2.3.22}$$

and

$$\begin{aligned}\left(\frac{w^2-u^2}{w^2-v^2}\right)_\xi = \frac{2}{(w^2-v^2)^2w^2(1-a\rho)}\Big[&uU_\xi\Big(\big[(n-1+2a\rho)w^2+\mu\alpha_0^2\rho(1-a\rho)\big](v^2-u^2)\\ &-2w^2(1-a\rho)(w^2-v^2)\Big)+vU_\eta\Big(\big[(n-1+2a\rho)w^2+\mu\alpha_0^2\rho(1-a\rho)\big](v^2-u^2)\\ &+2w^2(1-a\rho)(w^2-u^2)\Big)+2uw^2(1-a\rho)(w^2-v^2)\Big].\end{aligned} \tag{2.3.23}$$

With the help of equations (2.3.21)-(2.3.23), equation (2.3.13) reduces into the following form:

$$
\begin{aligned}
\partial_+\partial_- U = \frac{1}{w^2(w^2-u^2)(w^2-v^2)(1-a\rho)} & \Big[\Big(-u\big[(n-1)w^2+\mu\alpha_0^2\rho\big](v^2-u^2) \\
& + 2w^2(w^2-u^2)\Big)U_\xi^2 \\
& + \Big(2vw^2(u^2-w^2)+(3u^2-v^2)v\big[(n-1)w^2+\mu\alpha_0^2\rho\big]+\bar{R}\Big)U_\xi U_\eta \\
& + \Big(2uw^2(u^2-w^2)+2uv^2\big[(n-1)w^2+\mu\alpha_0^2\rho\big]+\Lambda_+\bar{R}+4uv^2\Big)U_\eta^2 \\
& - a\rho\big[\big(2uw^2+\mu\alpha_0^2\rho(v^2-u^2)+2w^2u(w^2-v^2)\big)U_\xi^2 \\
& + \big(2vw^2(u^2-v^2)+(3u^2-v^2)(-2w^2v+\mu\alpha_0^2\rho v)\big)U_\xi U_\eta \\
& \big(2uw^2(w^2-v^2)-4uv^2w^2+2uv^2\mu\alpha_0^2\rho\big)U_\eta^2\big]\Big] \\
& + \Big[\Big(\frac{-2v}{w^2-u^2}\Big)-(\partial_u\Lambda_-+\Lambda_+\partial_v\Lambda_-)\Big]+\Big(\frac{-2u}{w^2-u^2}\Big)U_\xi. \qquad (2.3.24)
\end{aligned}
$$

On simplification of $\partial_u\Lambda_- + \Lambda_+\partial_v\Lambda_-$, we get

$$
\partial_u\Lambda_- + \Lambda_+\partial_v\Lambda_- = \frac{2(u\Lambda_- - v)}{w^2-u^2}, \qquad (2.3.25)
$$

and using equation (2.3.25) into the last expression of equation (2.3.24), we get

$$
\Big[\Big(\frac{-2v}{w^2-u^2}\Big)-(\partial_u\Lambda_-+\Lambda_+\partial_v\Lambda_-)\Big]+\Big(\frac{-2u}{w^2-u^2}\Big)U_\xi = N_2\partial_- U, \qquad (2.3.26)
$$

where $N_2 = -\frac{2u}{w^2-u^2}$. We may factorize the quadratic form of equation (2.3.24) in the direction of Λ_-. Then, we have

$$
\begin{aligned}
\partial_+\partial_- U = \frac{1}{w^2(w^2-u^2)(w^2-v^2)(1-a\rho)} & \Big[\Big(-u\big[(n-1)w^2+\mu\alpha_0^2\rho\big](v^2-u^2) \\
& + 2w^2u(w^2-v^2)\Big)(U_\xi+\alpha_2 U_\eta) - a\rho\big(2uw^2+\mu\alpha_0^2\rho(v^2-u^2) \\
& + 2w^2u(w^2-v^2)\big)(U_\xi+\beta_2 U_\eta)\Big]\partial_- U + N_2\partial_- U, \qquad (2.3.27)
\end{aligned}
$$

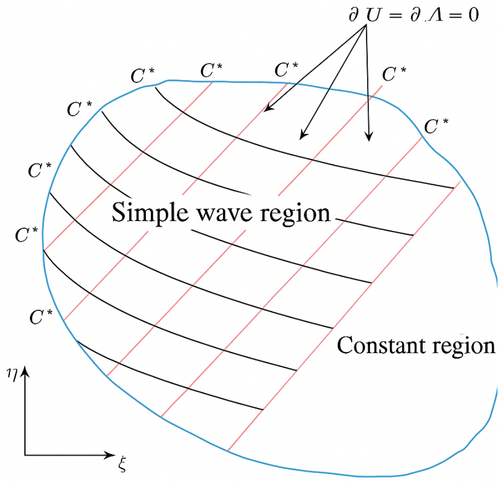

FIGURE 2.1: Flow adjacent to a constant state is simple.

where

$$\alpha_2 = \frac{2uw^2(w^2-v^2) + 2uv^2\big[(n-1)w^2 + \mu\alpha_0^2\rho\big] + \Lambda_+\bar{R} + 4uv^2}{\Lambda_-\Big(-u\big[(n-1)w^2 + \mu\alpha_0^2\rho\big](v^2-u^2) + 2w^2(w^2-v^2)\Big)},$$

and

$$\beta_2 = \frac{2uw^2(w^2-v^2) - 4uv^2w^2 + 2uv^2\mu\alpha_0^2\rho}{\Lambda_-\big(2uw^2 + \mu\alpha_0^2\rho(v^2-u^2) + 2w^2u(w^2-v^2)\big)}.$$

Therefore, equation (2.3.27) can be written as

$$\partial_+\partial_- U = N\partial_- U,$$

where $N = N_1 + N_2$, and

$$\begin{aligned} N_1 = &\Big(-u\big[(n-1)w^2 + \mu\alpha_0^2\rho\big](v^2-u^2) + 2w^2u(w^2-v^2)\Big)(U_\xi + \alpha_2 U_\eta) \\ &- a\rho\big(2uw^2 + \mu\alpha_0^2\rho(v^2-u^2) + 2w^2u(w^2-v^2)\big)(U_\xi + \beta_2 U_\eta). \end{aligned}$$

Similarly, we can show that

$$\partial_-\partial_+U = \overline{N}\partial_+U,$$

for some factor $\overline{N}$. Therefore, using the above result, we have the following theorem.

Theorem 2.3.1. *In a 2D pseudo-steady MHD system characterized by isentropic and irrotational flow with a non-ideal gas, if the flow occurs in a self-similar plane adjacent to a region of constant state, it must constitute a simple wave. Additionally, the flow variables (U, V, w) will remain constant. As a result, the corresponding family of wave characteristics will form straight lines (See Fig. 2.1) [35].*

2.4 Application of Characteristic Decomposition - Full MHD 2D system

The equation of continuity and equation of motion for a full MHD system are presented as

$$(U-\xi)\rho_\xi + (V-\eta)\rho_\eta + \rho\,(U_\xi + V_\eta) = 0, \tag{2.4.1}$$

$$(U-\xi)U_\xi + (V-\eta)U_\eta + \frac{p_\rho + \mu H H_\rho}{\rho}\rho_\xi = 0, \tag{2.4.2}$$

$$(U-\xi)V_\xi + (V-\eta)V_\eta + \frac{p_\rho + \mu H H_\rho}{\rho}\rho_\eta = 0, \tag{2.4.3}$$

$$(U-\xi)e_\xi + (V-\eta)e_\eta + \frac{p}{\rho}\rho_\xi = 0. \tag{2.4.4}$$

Here, e is the internal energy and is given by

$$e(\rho) = \frac{\mathcal{A}}{n-1}\Big(\frac{\rho}{1-a\rho}\Big)^{n-1} = \frac{1-a\rho}{n-1}\Big(\frac{p}{\rho}\Big),$$

where $n-1>0$. Now we define pseudo-flow directions $\partial_S := (U-\xi)\partial_\xi + (V-\eta)\partial_\eta$, which are opposed to the other two characteristic directions. In the self-similar plane, equations (2.4.1)-(2.4.4) take the form

$$\partial_S \rho + \rho\,(U_\xi + V_\eta) = 0,$$

$$\partial_S U + \frac{p_\rho + \mu H H_\rho}{\rho}\rho_\xi = 0,$$

$$\partial_S V + \frac{p_\rho + \mu H H_\rho}{\rho}\rho_\eta = 0,$$

$$\frac{1}{p\rho}\partial_S p + U_\xi + V_\eta = 0.$$

It is easy to verify that the entropy $S = \frac{p}{\left(\frac{\rho}{1-a\rho}\right)^n}$ is constant along with pseudo flow directions, that is,

$$\partial_S\left(\frac{p}{\left(\frac{\rho}{1-a\rho}\right)^n}\right) = 0.$$

Therefore, the flow along pseudo-flow lines is isentropic. Using the continuity and the momentum equations of a system of equations (2.1.3) and by denoting the vorticity as $\omega = V_x - U_y$, we get

$$\omega_t + (V\omega)_y + (U\omega)_x + \left(\frac{p_y + \mu H H_\rho \rho_y}{\rho}\right)_x - \left(\frac{p_x + \mu H H_\rho \rho_x}{\rho}\right)_y = 0.$$

In the self-similar plane, vorticity satisfies

$$\partial_S\left(\frac{\omega}{\rho}\right) = 0. \tag{2.4.5}$$

Therefore, the vorticity vanishes in a region where pseudo-flow characteristics arrive from a constant state. Hence, the region is isentropic and free from rotational flow. Therefore, applying the facts of the pseudo-steady case, we can state the following theorem.

Theorem 2.4.1. *A wave must be a simple wave for an isentropic and irrotational flow governed by a 2D full MHD system with a non-ideal gas in the self-similar plane* *adjacent to a region of the constant state. Also, the flow variables* (U, V, w, p, ρ) *will be constant, and the family of wave characteristics will be straight lines.*

2.5 Conclusion

This investigation is centered on finding the simple waves and the existence of the characteristic decomposition of the 2D compressible flow in a non-ideal MHD system. In this study, we used characteristic decomposition to elucidate the presence of simple waves within a two-dimensional compressible flow governed by a non-ideal magnetohydrodynamics system. We extended this analysis to steady and pseudo-steady states, showcasing the coexistence of simple waves alongside regions of constant state within the system. Additionally, we demonstrated the applicability of characteristic decomposition in pseudo-steady states, affirming the existence of simple waves within the full magnetohydrodynamics system by enforcing the constancy of vorticity and entropy along pseudo-flow characteristics. The phenomenal observations obtained from the current investigation can be condensed as follows:

- In a 2D steady MHD system with a non-ideal gas, any hyperbolic state adjacent to a constant state must be a simple wave.
- In a 2D pseudo-steady MHD system characterized by isentropic and irrotational flow with a non-ideal gas, if the flow occurs in a self-similar plane adjacent to a region of constant state, it must constitute a simple wave.
- A wave must be a simple wave for an isentropic and rotational flow governed by a 2D full MHD system with a non-ideal gas in the self-similar plane adjacent to a region of the constant state.

***